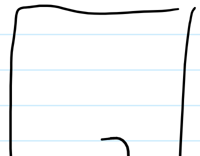
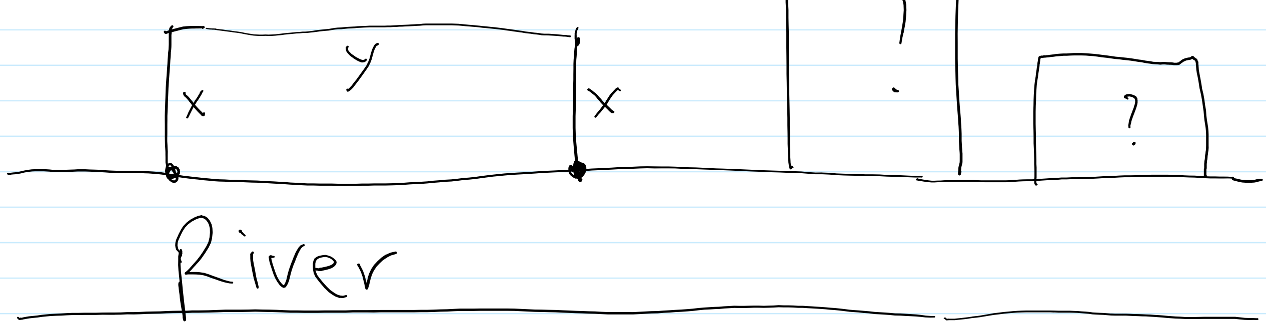
Problem 1: A farmer has 2400 m of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area?





Let x be one side of the field, and y be the other side (facing the river).

Area $= \boxed{A = x \cdot y = x(2400 - 2x)}$

Constraint: $2x + y = 2400$

$y = 2400 - 2x$

Need to find the global max for $A(x)$ on the interval $[0, 1200]$. $\left(\begin{matrix} x > 0 \\ y > 0 \end{matrix}\right)$

(i) Compute all crit. points of $A(x)$

(ii) Evaluate $A(x)$ at all crit. points and the end points.

(iii) Choose the largest value.

(i) $A'(x) = (2400x - 2x^2)' = 2400 - 4x = 0$

$x = \frac{2400}{4} = 600$

(ii) $A(0) = A(1200) = 0$

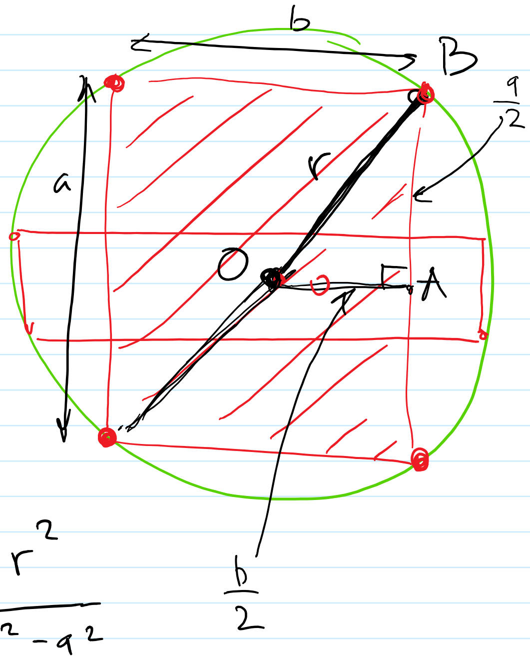
$A(600) = 2400 \cdot 600 - 2 \cdot 600^2 > 0$

This is the global max.

← This is the global max.

Answer: 720000

Problem 2: Find the dimensions of the rectangle of largest area that can be inscribed in a circle of radius r.



Let a, b be the sides of the inscribed rectangle.

Area = $A = a \cdot b$.

From the triangle OAB we have

$$\left(\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2 = r^2 \Rightarrow a^2 + b^2 = 4r^2$$

$\frac{a^2}{4}$

$$b = \sqrt{4r^2 - a^2}$$

So $\boxed{A(a) = a \cdot \sqrt{4r^2 - a^2}}$

Need to find the global max of $A(a)$.

To simplify calculations, we consider the function $f = A^2$ instead.

$$f(a) = a^2(4r^2 - a^2) = 4r^2 \cdot a^2 - a^4$$

$$f'(a) = 8r^2a - 4a^3 = 4a(2r^2 - a^2) = 0$$

$a = 0$ $\quad$ $2r^2 = a^2$

$a = r\sqrt{2}$

$\cancel{a = -r\sqrt{2}}$

Interval $a \in [0, 2r]$

$A(0) = 0$, $A(2r) = 0$

$$A(\sqrt{2} \cdot r) = \sqrt{2} \cdot r \cdot \sqrt{4r^2 - (\sqrt{2}r)^2} = \sqrt{2} \cdot r \cdot \sqrt{2}r = 2r^2 > 0$$

$2r^2$

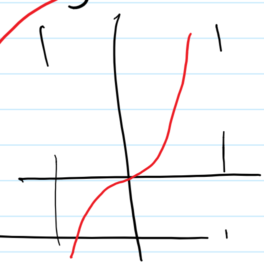
Global max.

The dimensions of the inscribed rectangle

The dimensions of the inscribed rectangle with the largest area are:

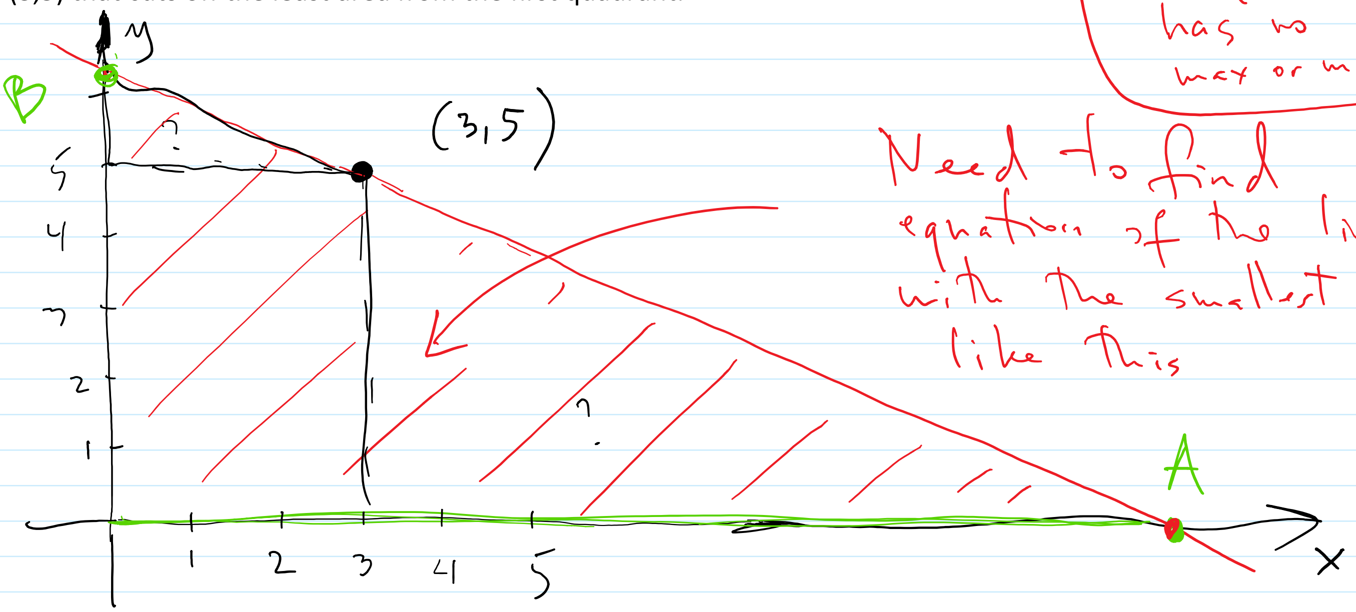
$$a = \sqrt{2} \cdot r$$
$$b = \sqrt{2} \cdot r$$

(square).



$y = \tan x$ on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ has no max or min

Problem 3: Find an equation of the line through the point (3,5) that cuts off the least area from the first quadrant.



Need to find equation of the line with the smallest Δ like this

Idea: find equation of the line.

$$y = mx + b$$

m – variable, find b from the fact that the line passes through (3,5)

$$y - y_0 = m(x - x_0) \qquad (x_0, y_0) = (3,5)$$
$$y - 5 = m(x - 3)$$
$$\boxed{y = m \cdot x + (5 - 3m)}$$

To find A: $y = 0 \Rightarrow mx + 5 - 3m = 0$ $\quad \boxed{x = \frac{3m-5}{m}}$

$$A\left(\frac{3m-5}{m}, 0\right)$$

To find B: $x = 0 \Rightarrow \boxed{y = 5 - 3m}$

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$B(0, 5-3m)$

Area of Δ: $\frac{1}{2}|OA| \cdot |OB| = \frac{1}{2}\left(\frac{3m-5}{m}\right) \cdot (5-3m)$

$$\boxed{A(m) = \frac{1}{2}\left(\frac{3m-5}{m}\right)(5-3m)} \quad \ldots$$

Problem 4: Show that of all the rectangles with a given area, the one with smallest perimeter is a square.