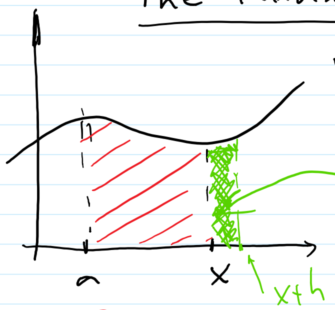
The Fundamental Theorem of Calculus. (5.3)



$$g(x) := \int_a^x f(t)\,dt$$

g is a function of x

? What is derivative of $g(x)$?

$$g'(x) = \lim_{h\to 0} \frac{g(x+h)-g(x)}{h}. \quad \text{So for } h \ll 1.$$

$$g'(x) \approx \frac{g(x+h)-g(x)}{h}$$

green area $\approx$ area of rectangle $= f(x)\cdot h$

So $g(x+h) - g(x) \approx f(x)\cdot h$ or

$$\frac{g(x+h)-g(x)}{h} \approx f(x).$$

So maybe $g'(x) = f(x)$?

Fundamental Theorem of Calculus (Part I):

Suppose $f(x)$ is continuous on $[a,b]$. Then the function

$$\boxed{g(x) = \int_a^x f(t)\,dt}$$

is continuous on $[a,b]$, differentiable on (a,b), and $g'(x) = f(x)$.

Proof: We want to calculate $g'(x) = \lim_{h\to 0} \frac{g(x+h)-g(x)}{h}$

Start with $g(x+h) - g(x) = \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt \;(=)$

$$\left[\int_a^x f(t)\,dt + \int_x^{x+h} f(t)\,dt = \int_a^{x+h} f(t)\,dt\right]$$

(†)

$(=) \int^{x+h} f(t)\,dt \;\Rightarrow\; \frac{g(x+h)-g(x)}{h} = \frac{1}{h}\int^{x+h} f(t)\,dt$

$$\Leftrightarrow \int_x^{x+h} f(t)\,dt \Rightarrow \boxed{\frac{g(x+h)-g(x)}{h} = \frac{1}{h}\int_x^{x+h} f(t)\,dt}$$

$\boxed{h>0}$

Since $f(x)$ is continuous on $[x, x+h]$, By the Extreme Value Theorem, $\exists\ m =$ global min of f on $[x, x+h]$

$\exists\ M =$ global max.

Say $m = f(u)$, $M = f(v)$, $u, v \in [x, x+h]$.

Then $m \le f(x) \le M \Rightarrow$

Property of def. $\int$: if $f \le M$, then $\int_a^b f(t)\,dt \le M(b-a)$

$$m \cdot h \le \int_x^{x+h} f(t)\,dt \le M \cdot h$$

$$\Big\Downarrow \quad m = f(u),\ M = f(v)$$

$$f(u)\cdot h \le \int_x^{x+h} f(t)\,dt \le f(v)\,h$$

$$\Big\Downarrow \text{ divide by } h$$

$$f(u) \le \frac{1}{h}\int_x^{x+h} f(t)\,dt \le f(v)$$

$$\Big\Downarrow (+)$$

$$f(u) \le \frac{g(x+h)-g(x)}{h} \le f(v)$$

What happens when $h \to 0$? We will have

$$\lim_{h\to 0} \frac{g(x+h)-g(x)}{h} = f(x). \quad \square$$

Another notation: $\boxed{\frac{d}{dt}\left[\int_a^x f(t)\,dt\right] = f(x)}$

Examples:

① find $g'(x)$ if $g(x) = \int_0^x \frac{dt}{\sqrt{1+t^2}}$.

Solution: The function $\frac{1}{\sqrt{1+x^2}}$ is continuous on $\mathbb{R}$, so we may apply the FTC. We have

$$g'(x) = \frac{1}{\sqrt{1+x^2}}.$$

② $S(x) = \int_0^x \cos(t^2+1)\,dt$. Find $S''(x)$.

Solution: The integrand is a continuous function so the FTC applies:

$$S'(x) = \cos(x^2+1)$$

$$S''(x) = -\sin(x^2+1)\cdot 2x.$$

$x = t$

define x to be t

equation look for particular value of x

$x := t$

③ $g(x) := \int_0^{x^4} \tan s\,ds$. Find $g'(x)$.

$=$ | need to use the Chain Rule $u = x^4$, $\frac{du}{dx} = 4x^3$. |

$$g'(x) = \frac{d}{du}\left[\int_0^u \tan s\,ds\right]\cdot\frac{du}{dx} \overset{FTC}{=} \tan u\cdot 4x^3$$

$$= (\tan x^4)\cdot 4x^3.$$

(?) For what values of x is this calculation valid?

Well, To apply the FTC we need the function under the integral sign to be continuous. $\tan s$ is not defined at $s = \frac{\pi}{2}$, so we cannot apply FTC if $x \geq \frac{\pi}{2}$. So the calculation is valid for $0 < x < \frac{\pi}{2}$.