(5.5) Substitution Rule

$$\int f(g(x))\,g'(x)\,dx = F(g(x)), \text{ where } F' = f.$$

Pf: $[F(g(x))]' = F'(g(x))\cdot g'(x) = f(g(x))\,g'(x)$

Examples:

① $\int x^3 \sin(x^4+1)\,dx = \left|\begin{array}{l} y = x^4+1 \\ dy = (x^4+1)'\,dx \\ \quad = 4x^3\,dx \end{array} \quad x^3\,dx = \frac{1}{4}\,dy \right|$

$= \int \sin y \cdot \frac{1}{4}\,dy = \frac{1}{4}\int \sin y\,dy = -\frac{1}{4}\cos y + C$

$= -\frac{1}{4}\cos(x^4+1) + C.$

② $\int \sqrt{\underbrace{3-2x}_{=u}}\,dx = \left|\begin{array}{l} u = 3-2x \\ du = -2\,dx \\ dx = -\frac{1}{2}\,du \end{array}\right| = \int \sqrt{u}\left(-\frac{1}{2}\right)du$

$= -\frac{1}{2}\int u^{\frac{1}{2}}\,du = -\frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{3}u^{\frac{3}{2}} + C = -\frac{1}{3}u^{\frac{3}{2}} + C.$

③ $\int \sin 4x\,dx = \left|\begin{array}{l} t = 4x \\ dt = 4\,dx \end{array}\right| = \int \sin t\,\frac{1}{4}\,dt$

$= \frac{1}{4}\cdot(-\cos t) + C = -\frac{1}{4}\cos(4x) + C.$

④ $\int \underbrace{\sqrt{1+x^2}}_{=\sqrt{y}}\, x^5\,dx = \left|\begin{array}{l} y = 1+x^2 \\ dy = 2x\,dx \end{array}\right| \quad x^2 = y-1 \quad x^4 = (y-1)^2 \quad \text{⊜}$

$x^5\,dx = \underbrace{(x\,dx)}_{=\frac{dy}{2}}\cdot\underbrace{x^4}_{=(y-1)^2}$

⊜ $\frac{1}{2}\int \sqrt{y}\cdot(y-1)^2\,dy = \frac{1}{2}\int u^{\frac{1}{2}}(u^2 - 2u + 1)\,du$

$$\ldots = \frac{1}{2}\int y^{\frac12}\left(y^2-2y+1\right)dy$$

$$= \frac12\left[\int y^{\frac52}\,dy - 2\int y^{\frac32}\,dy + \int y^{\frac12}\,dy\right]$$

$$= \frac12\cdot\frac27\cdot y^{\frac72} - \frac25 y^{\frac52} + \frac12\,\frac23 y^{\frac32} + C$$

$$= \frac17 y^{\frac72} - \frac25 y^{\frac52} + \frac13 y^{\frac32} + C.$$

5. $$\int \tan x\,dx = \int \frac{\overbrace{\sin x\,dx}^{=-dv}}{\underbrace{\cos x}_{v}} = \left| \begin{array}{l} \cos x = v \\ -\sin x\,dx = dv \end{array} \right|$$

$$= -\int \frac{dv}{v} = -\ln|v| + C = -\ln|\cos x| + C.$$

Substitution for definite integrals

$$\boxed{\int_a^b f(g(x))\,g'(x)\,dx = \left| \begin{array}{lll} u = g(x) & a \to g(a) \\ & b \to g(b) \end{array} \right| = \int_{g(a)}^{g(b)} f(u)\,du}$$

Example:

1'. $$\int_0^1 x^3 \sin(x^4+1)\,dx = -\frac14 \cos y \Big|_1^2 = -\frac14 \cos 2 + \frac14 \cos 1$$

$y = x^4+1$

$0 \to 0^4+1 = 1$

$1 \to 1^4+1 = 2$

$$= -\frac14 \int_1^2 \sin y\,dy =$$