November 30, 2016 1:31 PM

Substitution Rule (Cont'd)

$$\boxed{\int_a^b f(g(x))\,g'(x) = \int_{g(a)}^{g(b)} f(u)\,du \qquad \left(\begin{array}{c}\text{substitution}\\ u=g(x)\end{array}\right)}$$

Example: ①

$$\int_1^2 \frac{dx}{(3-5x)^2} = \left|\begin{array}{l} 3-5x=u \\ -5\,dx = du \\ 1 \longrightarrow 3-5(1)=-2 \\ 2 \longrightarrow 3-5(2)=-7 \end{array}\right|$$

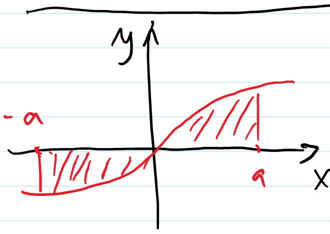
$$= \int_{-2}^{-7} \frac{-\frac{1}{5}\,du}{u^2} = -\frac{1}{5}\int_{-2}^{-7} u^{-2}\,du = -\frac{1}{5}\left(-u^{-1}\right)\Big|_{-2}^{-7}$$

$$= \frac{1}{5u}\Big|_{-2}^{-7} = -\frac{1}{35} - \left(-\frac{1}{10}\right) = \frac{1}{10} - \frac{1}{35} = \frac{1}{14}.$$

②

$$\int_1^e \frac{\ln x}{x}\,dx = \left|\begin{array}{l} \cancel{\frac{1}{x}=y \quad -\frac{dx}{x^2}=dy} \\ \ln x = y \quad \frac{dx}{x} = dy \\ x=1 \to y=\ln 1 = 0 \\ x=e \to y = \ln e = 1 \end{array}\right| = \int_0^1 y\,dy$$

$$= \frac{1}{2}y^2\Big|_0^1 = \frac{1}{2}.$$



$\int_{-a}^{a} f(x)\,dx = 0$ if f is odd

Prop: If $f(x)$ is an odd function, i.e., $\boxed{f(-x) = -f(x)}$, (*)
Then $\int_{-a}^{a} f(x)\,dx = 0$ for any $a>0$.

Pf: $\int^{a} f(x)\,dx = \int^{0} f(x)\,dx + \int^{a} f(x)\,dx = 0$

Pf: $\displaystyle\int_{-a}^{a} f(x)\,dx = \underbrace{\int_{-a}^{0} f(x)\,dx}_{} + \int_{0}^{a} f(x)\,dx = 0.$

$$\int_{-a}^{0} f(x)\,dx = \left|\begin{array}{l} t=-x \\ dt=-dx \\ -a \longrightarrow a \\ 0 \longrightarrow 0 \end{array}\right| = \int_{a}^{0} f(-t)(-dt)$$

$$= \int_{0}^{a} f(-t)\,dt \overset{(*)}{=} -\int_{0}^{a} f(t)\,dt \qquad \square$$

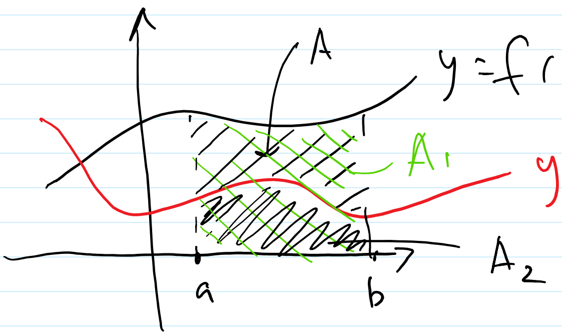
Similarly, if f is even, i.e., $f(-x)=f(x)$, then $\boxed{\displaystyle\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx}$

$$a=b$$
$$a^2=ab$$
$$a^2-b^2=ab-b^2$$
$$\underset{=0}{\cancel{(a-b)}}(a+b)=b\underset{=0}{\cancel{(a-b)}}$$
$$a+b=b$$
$$2\cancel{b}=\cancel{b}$$
$$2=1.$$

$1\cdot 0 = 2\cdot 0$

$1 \neq 2$

6.1 Areas between the curves.



$A = ? = A_1 - A_2$

$\displaystyle\int_a^b f(x)\,dx = A_1$

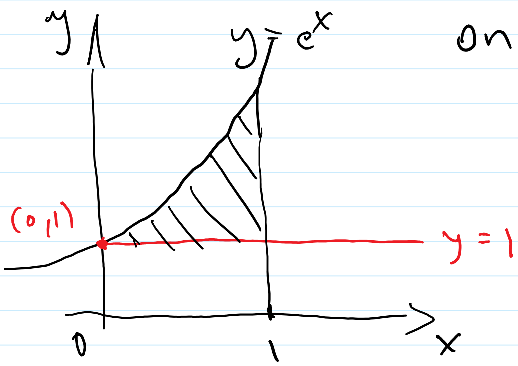
$\displaystyle\int_a^b g(x)\,dx = A_2$

So $\qquad A = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

$$\boxed{A = \int_a^b [f(x) - g(x)]\,dx}$$

f = "upper graph"
g = "lower graph".

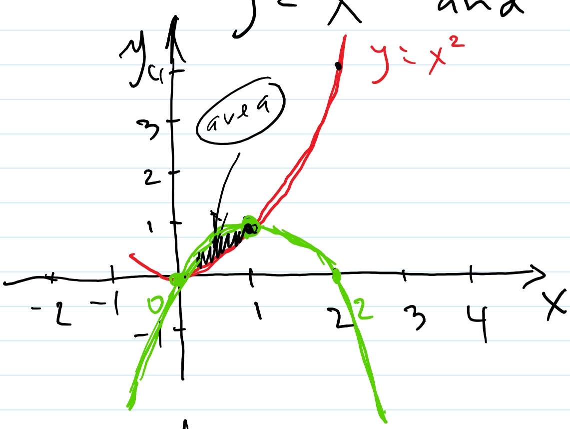
Examples: (3) Find the area between the graphs of $y = e^x$ and $y = 1$ on $[0, 1]$



$$\int_0^1 (e^x - 1)\,dx = e^x - x \Big|_0^1$$
$$= (e - 1) - (e^0 - 0)$$
$$= e - 2.$$

(4) Find the area between the curves $y = x^2$ and $y = 2x - x^2$.

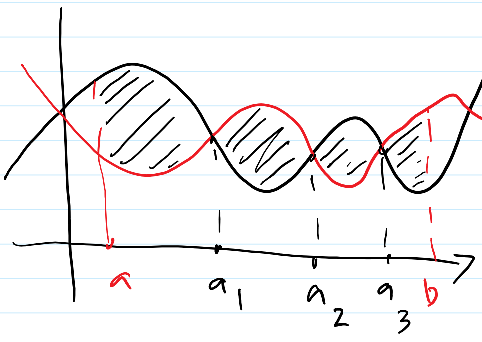
$x(2-x)$, $y(1) = 1$



$y = x^2$ and $y = 2x - x^2$ intersect at $(0, 0)$ and $(1, 1)$

So the area equals $\int_0^1 [(2x - x^2) - x^2]\,dx$

$$= \int_0^1 (2x - 2x^2)\,dx = \left(x^2 - \frac{2}{3}x^3\right)\Big|_0^1 = 1 - \frac{2}{3} = \frac{1}{3}.$$

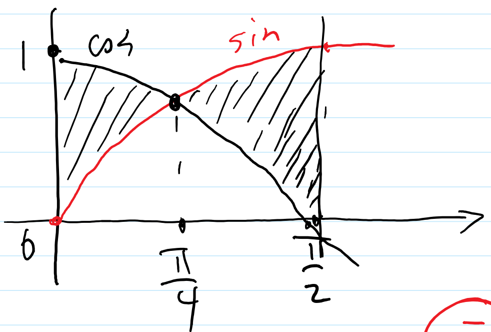


Area between f and ... = ?
from a to b

$$\text{Area} = \int_a^{a_1} (f-g)\,dx + \int_{a_1}^{a_2} (g-f)\,dx + \int_{a_2}^{a_3} (f-g)\,dx + \int_{a_3}^{b} (g-f)\,dx$$

$$\boxed{\text{Area} = \int_a^b |f-g|\,dx}$$

(5) Find area between sin x and cos x for x from 0 to $\frac{\pi}{2}$.



$$\text{Area} = \int_0^{\frac{\pi}{2}} |\sin x - \cos x|\,dx =$$

$\sin x = \cos x \Rightarrow x = \frac{\pi}{4}$

$$= \int_0^{\frac{\pi}{4}} (\cos x - \sin x)\,dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\sin x - \cos x)\,dx$$

$$= \sin x + \cos x \Big|_0^{\frac{\pi}{4}} + (-\cos x - \sin x)\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) - 1 + \left(-1 - \left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right)\right).$$