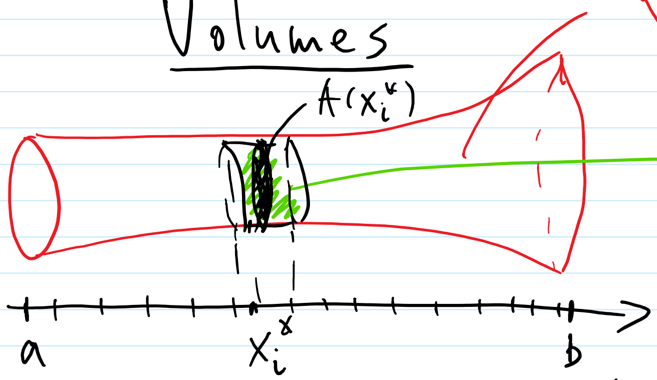
December 5, 2016 1:32 PM

Volumes

V = volume?



treat this as a cylinder

$A(x_i^*)$ = area of the section $\boxed{x = x_i^*}$

divide $[a, b]$ into n subintervals of length Δx

choose point x_i^* in every subinterval $[x_i, x_{i+1}]$

Volume of the cylinder $= A(x_i^*) \cdot \Delta x$

$$\boxed{\Delta x = \frac{b-a}{n}}$$

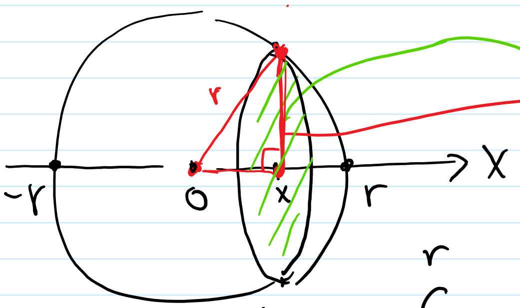
$$\text{Volume of the solid} \approx \underbrace{\sum_{i=0}^{n-1} A(x_i^*)\Delta x}_{\text{integral sum}}$$

$$\boxed{\text{Volume} = \lim_{n\to\infty} \sum_{i=0}^{n} A(x_i^*)\Delta x = \int_a^b A(x)\,dx}$$

$\boxed{A(x) = \text{cross-section area}}$

Examples:

① Find the volume of a sphere of radius r.



$\sqrt{r^2 - x^2}$

$A(x) = \pi\left(\sqrt{r^2 - x^2}\right)^2$ area of cross-section

$$\text{volume} = \int_{-r}^{r} \pi(r^2 - x^2)\,dx = \pi\left(r^2x - \tfrac{1}{3}x^3\right)\Big|^{r}$$

section

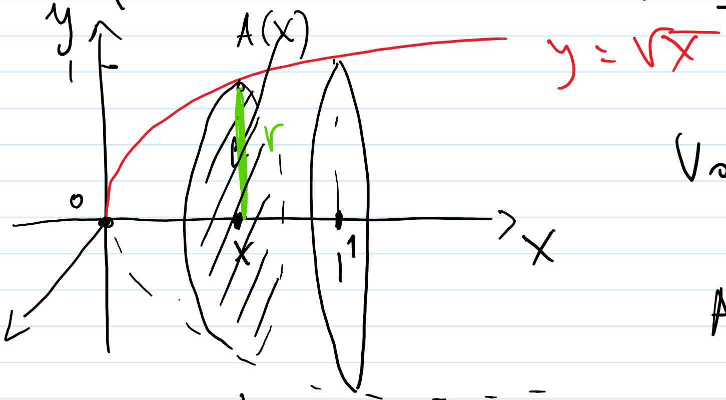
$$\text{volume} = \int_{-r}^{r} \pi (r^2 - x^2)\,dx = \pi \left(r^2 x - \frac{1}{3}x^3 \right) \Big|_{-r}^{r}$$

$$= \pi \left(r^3 - \frac{1}{3} r^3 \right) - \pi \left(\underbrace{r^2(-r) - \frac{1}{3}(-r)^3}_{-r^3 + \frac{1}{3}r^3 \;=\; -\frac{2}{3}r^3} \right)$$

$$= \pi \frac{2}{3} r^3 + \pi \frac{2}{3} r^3 \Rightarrow$$

$$\boxed{\text{Vol} = \frac{4}{3}\pi r^3}$$

② Find the volume of the solid obtained by rotating about the x-axis the region under the graph of $y = \sqrt{x}$ from $x = 0$ to $x = 1$.



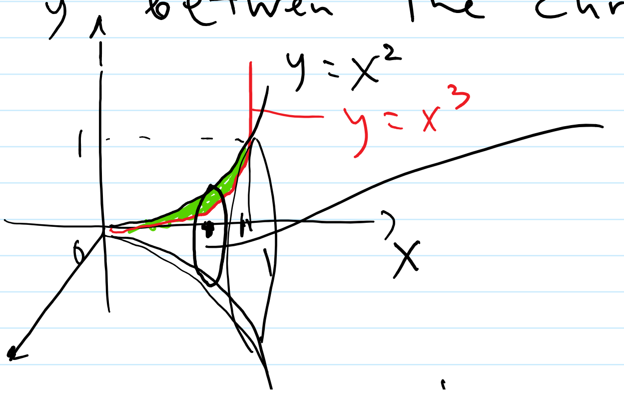
$$\text{Vol} = \int_0^1 A(x)\,dx =$$

$$A(x) = \pi r^2 = \pi(\sqrt{x})^2 = \pi x$$

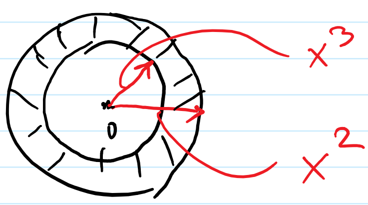
$$= \int_0^1 \pi x\,dx = \pi \frac{x^2}{2} \Big|_0^1 = \frac{\pi}{2}.$$

"rotation $\Rightarrow$ cross-section is a circle"

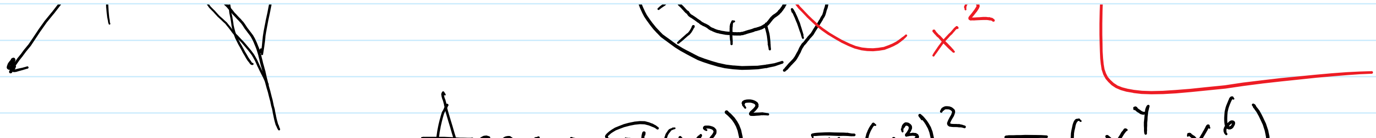
③ Same question as in Ex 2 for the area between the curves $y = x^2$ and $y = x^3$.



cross section:



If $0 < x < 1$ then $x^3 < x^2$



$$\text{Area} = \pi (x^2)^2 - \pi (x^3)^2 = \pi (x^4 - x^6)$$

So $\text{Volume} = \int_0^1 \pi (x^4 - x^6)\, dx = \pi \left(\frac{x^5}{5} - \frac{x^7}{7} \right) \Big|_0^1$

$$= \pi \left(\frac{1}{5} - \frac{1}{7} \right) = \frac{2\pi}{35}.$$

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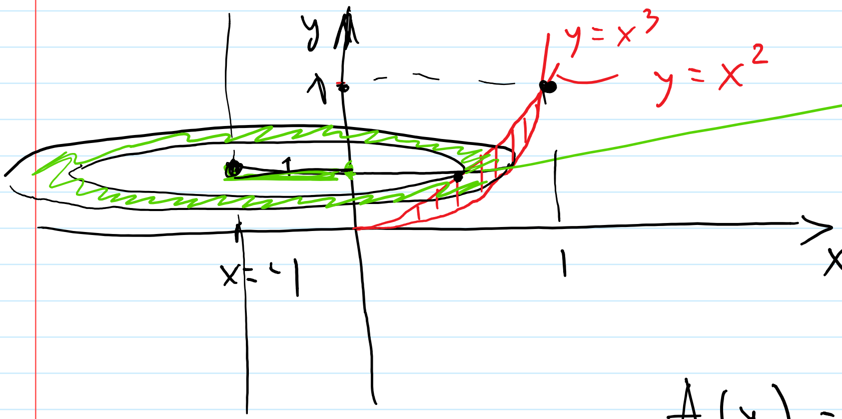
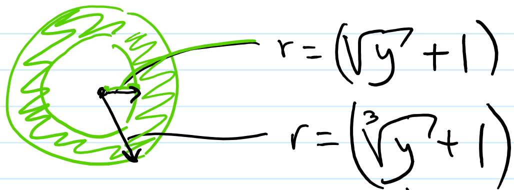
(4) Same as in example (3) but rotate about the line $x = -1$.



$A(y)$ - area of cross section.



$$A(y) = \pi \left((\sqrt[3]{y} + 1)^2 - (\sqrt{y} + 1)^2 \right) \quad y > 0$$

$$= \pi \left(y^{\frac{2}{3}} + 2 y^{\frac{1}{3}} + \cancel{1} - y - 2\sqrt{y} - \cancel{1} \right)$$

$$= \pi \left(y^{\frac{2}{3}} + 2 y^{\frac{1}{3}} - y - 2\sqrt{y} \right)$$

$$\text{Volume} = \pi \int_0^1 \left(y^{\frac{2}{3}} + 2 y^{\frac{1}{3}} - y - 2 y^{\frac{1}{2}} \right) dy$$

$$= \pi \left[\frac{3}{5} y^{\frac{5}{3}} + \frac{3}{2} y^{\frac{4}{3}} - \frac{1}{2} y^2 - \frac{4}{3} y^{\frac{3}{2}} \right] \Big|_0^1$$

$$= \pi \left[\frac{3}{5} + \frac{3}{2} - \frac{1}{2} - \frac{4}{3} \right].$$