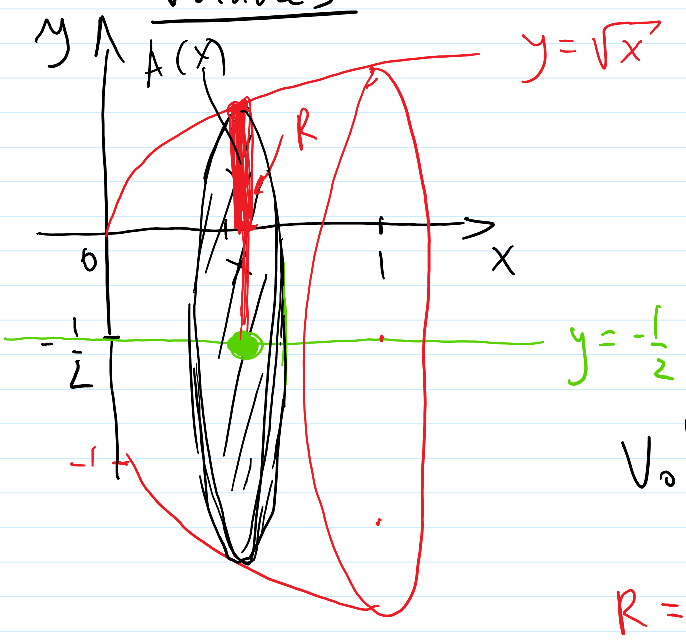
December 7, 2016 1:30 PM

Volumes



from 0 to 1

Example 1 Find the volume of a solid obtained by rotating the graph $y=\sqrt{x}$ about the line $y=-\frac{1}{2}$.

$$\text{Volume} = \int_0^1 A(x)\,dx$$

$$R = \frac{1}{2}+\sqrt{x} \Rightarrow A(x) = \pi R^2 = \pi\left(\frac{1}{2}+\sqrt{x}\right)^2$$

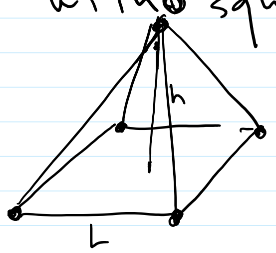
$$\Rightarrow \text{Vol} = \pi\int_0^1 \left(\frac{1}{2}+\sqrt{x}\right)^2 dx = \pi\int_0^1 \left(\frac{1}{4}+\sqrt{x}+x\right)dx$$

$\sqrt{x} = x^{\frac{1}{2}}$

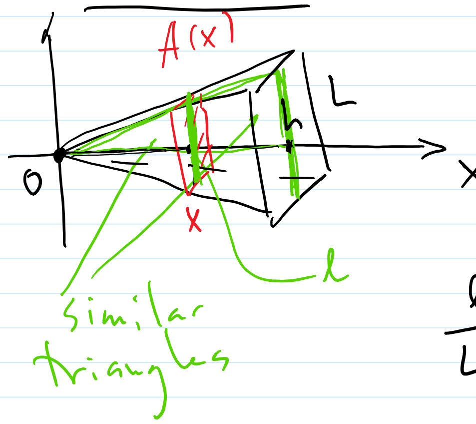
$$= \pi\left(\frac{1}{4}x+\frac{2}{3}x^{\frac{3}{2}}+\frac{1}{2}x^2\right)\Bigg|_0^1 = \pi\left(\frac{1}{4}+\frac{2}{3}+\frac{1}{2}\right)$$

$$\pi\,\frac{3+8+6}{12} = \frac{17}{12}\pi$$

Example 2 Find the area of a pyramid with square base of length L and height h.



Solution:



cross-section is a square.

$$\frac{l}{L} = \frac{x}{h} \Rightarrow \boxed{l = \frac{L}{h}\cdot x}$$

So the area of the cross-section at pt x is

$$A(x) = l^2 = \frac{L^2}{h^2}\cdot x^2.$$

$$A(x) = e^2 = \frac{L^2}{h^2} \cdot x^2.$$

So Volume of the pyramid $= \int_0^h A(x)\,dx$

$$= \int_0^h \frac{L^2}{h^2} x^2\,dx = \frac{L^2}{h^2} \cdot \frac{1}{3} x^3 \Big|_0^h = \frac{L^2}{h^2} \cdot \frac{1}{3} h^3 = \boxed{\frac{1}{3} L^2 h}$$

Material since the midterm:

- Derivatives of $\log x$, etc.

$$\frac{d}{dx}\left[\ln(\sin x)\right] = \frac{\cos x}{\sin x} = \cot x.$$

- related rates
- Max/Min Values: $\to$ abs max or min on $[a,b]$
  $\to$ local max/min (extr.) (related to crit. #s $f'(c)=0$).

- L'Hospital's Rule:

$$\lim_{\substack{x \to a \\ \text{or } x \to \pm\infty}} \frac{f(x)}{g(x)} = \left| \begin{matrix} f(a)=0 \\ g(a)=0 \end{matrix} \right| = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

$\frac{0}{0}$

e.g. $$\lim_{x \to 1^-} \left(\frac{x}{x-1} - \frac{1}{\ln x} \right) = \left| \underbrace{-\infty + \infty}_{\text{indet.}} \right| \;(=)$$

$\frac{1}{\ln x} \to -\infty \Rightarrow -\frac{1}{\ln x} \to \infty$ as $x \to 1^-$.

$\frac{x}{x-1} \to -\infty$ as $x \to 1^-$

$$(=) \lim_{x \to 1^-} \left[\frac{x \ln x - x + 1}{\ln x\,(x-1)} \right] = \left| \frac{0}{0} \right| \overset{\text{L'H.R}}{=}$$

$\lim \quad \ln x + \cancel{1} - \cancel{1} \quad \quad \lim \left[\ln x \right]$

$$(x\ln x)' = \ln x + x\cdot\frac{1}{x} = \ln x + 1$$

$$= \lim_{x\to 1^-} \frac{\ln x + \cancel{1} - \cancel{1}}{\frac{1}{x}(x-1) + \ln x} = \lim_{x\to 1^-}\left[\frac{\ln x}{\frac{(x-1)}{x} + \ln x}\right]$$

$$= \lim_{x\to 1^-}\left[\frac{x\ln x}{x\ln x + (x-1)}\right] = \left|\frac{0}{0}\right|$$

$$= \lim_{x\to 1^-}\left[\frac{\ln x + 1}{\ln x + 1 + 1}\right] = \lim_{x\to 1^-}\frac{\ln x + 1}{\ln x + 2}$$

$$= \frac{1}{2}.$$

- Optimization
- Antiderivatives
- Definite integrals, Indef. Int, FTC I, II

$$\boxed{\int_a^b f(x)\,dx = F(b) - F(a) \qquad F' = f}$$

Example: Prove that

$$\int_0^1 \underbrace{x\sqrt{1+x^2}}_{f}\,dx \le \int_0^1 \underbrace{\sqrt{1+x}}_{g}\,dx$$

Solution 1:

- $$\int_0^1 x\sqrt{1+x^2}\,dx = \left|\begin{array}{l} y = x^2+1 \\ dy = 2x\,dx \\ x=0 \to 1=y \\ x=1 \to y=2 \end{array}\right|$$

$$= \frac{1}{2}\int_1^2 \sqrt{y}\,dy = \frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{3}\cdot y^{\frac{3}{2}}\Big|_1^2 = \frac{1}{3}\left(\sqrt{8}-1\right).$$

$y^{\frac{1}{2}}$

- $$\int_0^1 \sqrt{1+x}\,dx = \left|\begin{array}{l} 1+x = y \\ dx = dy \end{array}\right| = \int_1^2 \sqrt{y}\,dy$$

$$= \frac{2}{3}y^{\frac{3}{2}}\Big|_1^2 = \frac{2}{3}\left(\sqrt{8}-1\right).$$

Solution 2 : $f = x\sqrt{1+x^2}$

$g = \sqrt{1+x}$.

Claim : $f \le g$ on $[0,1]$.

$x \le 1$, $1+x^2 \le 1+x$ So

$$f = \underbrace{x}_{\le 1}\sqrt{\underbrace{1+x^2}_{\le 1+x}} \le \sqrt{1+x} = g.$$

[So By the properties of definite integrals, if $f \le g$ on $[a,b]$, then $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$

$\blacksquare$

• Area, Volume.