Properties of limits

If $\lim_{x\to x_0} f(x) = A$, $\lim_{x\to x_0} g(x) = B$, Then

$$\lim_{x\to x_0} (f+g) = A+B$$

$$\lim_{x\to x_0} (f\cdot g) = A\cdot B$$

$$\lim_{x\to x_0} \left(\frac{f}{g}\right) = \frac{A}{B}, \quad B \neq 0.$$

$$\lim_{x\to x_0} (c\cdot f) = cA, \quad c \in \mathbb{R}$$

$$\lim_{x\to x_0} (f(x))^n = A^n$$

$$\lim_{x\to x_0} \sqrt[m]{f(x)} = \sqrt[m]{A}$$

Most basic example: $\boxed{\lim_{x\to x_0} x = x_0}$

Take any polynomial $P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$

$a_j \in \mathbb{R}$, $j = 0, \dots, n$

$\lim_{x\to x_0} P(x) = P(x_0)$, e.g. $\lim_{x\to 1} x^2 + 3x - 1 = 3$.

In the same way for any rational function

$R(x) = \frac{P(x)}{Q(x)}$, P, Q are polynomials

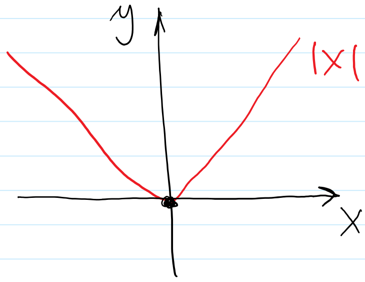
$$\lim_{x\to x_0} R(x) = R(x_0), \text{ if } Q(x_0) \neq 0.$$

Example 1: $\lim_{x\to 1} \frac{x^3-1}{x^2-1} = \lim_{x\to 1} \frac{x^2+x+1}{x+1} = \frac{3}{2}$

$x \to 1 \qquad x+1 \qquad -2$

$$\frac{x^3-1}{x^2-1} = \frac{\cancel{(x-1)}(x^2+x+1)}{\cancel{(x-1)}(x+1)} = \frac{x^2+x+1}{x+1}$$

Example 2: $\lim\limits_{x\to 0} |x| = ?$



$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

$$\left.\begin{array}{l} \lim\limits_{x\to 0^+} |x| = \lim\limits_{x\to 0^+} x = 0 \\ \lim\limits_{x\to 0^-} |x| = \lim\limits_{x\to 0^-} -x = 0 \end{array}\right\} \Rightarrow \lim_{x\to 0} |x| = 0.$$

$$\lim_{x\to 0} \frac{|x|}{x} = ?$$

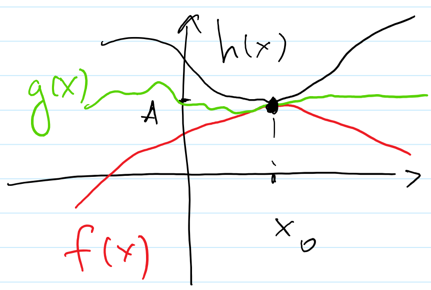
$$\left.\begin{array}{l} \lim\limits_{x\to 0^+} \frac{|x|}{x} = \lim\limits_{x\to 0^+} \frac{x}{x} = 1. \\ \lim\limits_{x\to 0^-} \frac{|x|}{x} = \lim\limits_{x\to 0^-} \frac{-x}{x} = -1 \end{array}\right\} \Rightarrow \lim_{x\to 0} \frac{|x|}{x} \text{ does not exist.}$$

Thm (Squeeze Theorem)

Suppose $f(x)$, $g(x)$, $h(x)$ satisfy $\boxed{\lim\limits_{x\to x_0} f(x) = \lim\limits_{x\to x_0} h(x) = A}$ and

$f(x) \leq g(x) \leq h(x)$ for all x.

Then $\lim\limits_{x\to x_0} g(x) = A$.



$f(x) \;|\; x_0$

Example: $\boxed{\lim\limits_{x\to 0} x \cdot \sin \frac{1}{x}}$ $\qquad \boxed{-1 \le \sin\frac{1}{x} \le 1}$

$\boxed{g = x \sin\frac{1}{x}, \quad f = -x, \quad h = x}$

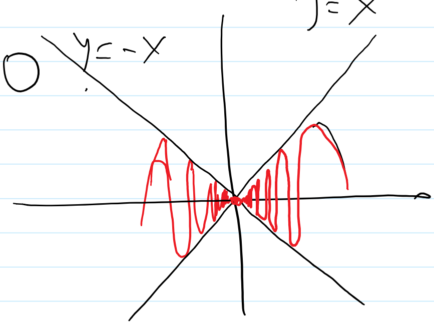
$$f \le g(x) \le h$$

for $x > 0$ $\longrightarrow$ $\boxed{-x \le x \sin\frac{1}{x} \le x}$

$$\lim_{x\to 0} -x = \lim_{x \to 0} x = 0$$

As in the Squeeze Theorem!!!

$$\xRightarrow{\text{Sq. Thm.}} \lim_{x\to 0} x \cdot \sin\frac{1}{x} = 0$$



If $x < 0$, then

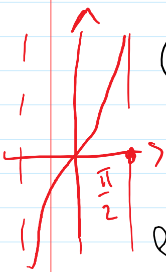
$$x \le x \sin\frac{1}{x} \le -x$$

Definition: A function $f(x)$ is called continuous at x_0, if

$$\boxed{\lim_{x\to x_0} f(x) = f(x_0)}$$

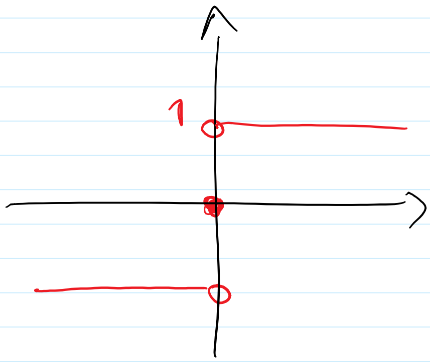
So we already know that any polynomial is a continuous function (at any point); any rational function is continuous at points where denominator does not vanish.

In fact, any exp. function, logarithm, any trig function is continuous on its domain.



e.g. $f(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$

$\boxed{\text{SIGNUM}}$



$\lim_{x \to 0} f(x) \neq 0 = f(0)$ f is not continuous at 0.

But f is continuous at all other points.

Def: $\lim_{x \to 0} f(x) = A$ if $\forall \varepsilon > 0 \; \exists \delta > 0$ such that $|x - x_0| < \delta \Rightarrow |f(x) - A| < \varepsilon$.