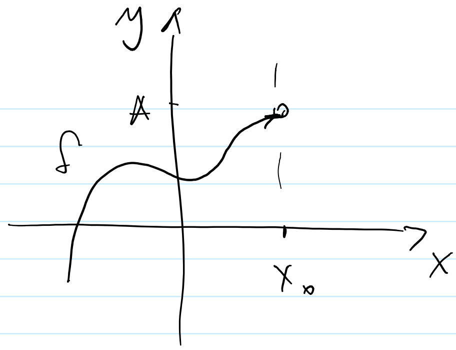
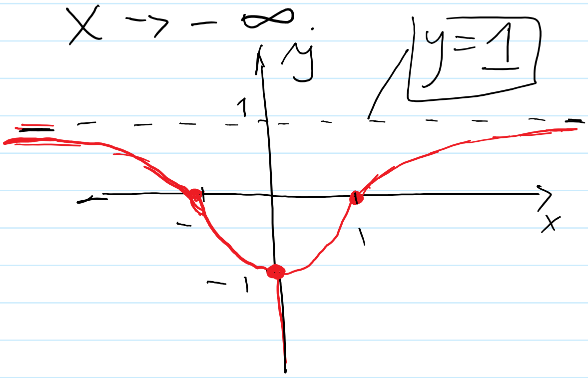
Limits (continued)

$$\lim_{x\to x_0} f(x) = A$$



What if $x \to \infty$ or $x \to -\infty$.

Example 1: $y = \dfrac{x^2-1}{x^2+1}$

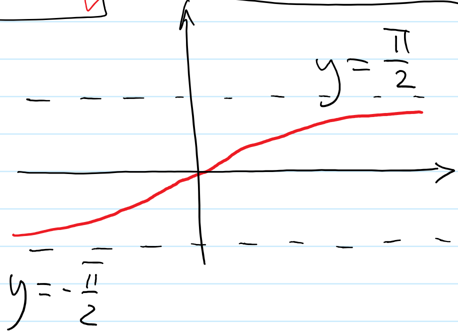


$$\lim_{x\to\infty} \frac{x^2-1}{x^2+1} = 1.$$

$$\lim_{x\to-\infty} \frac{x^2-1}{x^2+1} = 1.$$

Def: A function $f(x)$ has a horizontal asymptote $\boxed{y = L}$ if $\boxed{\lim_{x\to\infty} f(x) = L}$ or $\boxed{\lim_{x\to-\infty} f(x) = L}$

Example 2: $y = \arctan x$



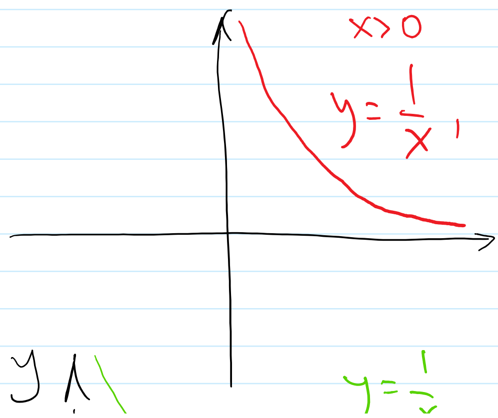
$$\lim_{x\to\infty} \tan^{-1} x = \frac{\pi}{2}$$

$$\lim_{x\to-\infty} \tan^{-1} x = -\frac{\pi}{2}$$

Consider, $y = \dfrac{1}{x^r}$, $r > 0$

$$\lim_{x\to\infty} \frac{1}{x^r} = \lim_{x\to\infty} \left(\frac{1}{x}\right)^r = 0$$

$$\boxed{\lim_{x\to\infty} \frac{1}{x^r} = 0, \quad r > 0}$$



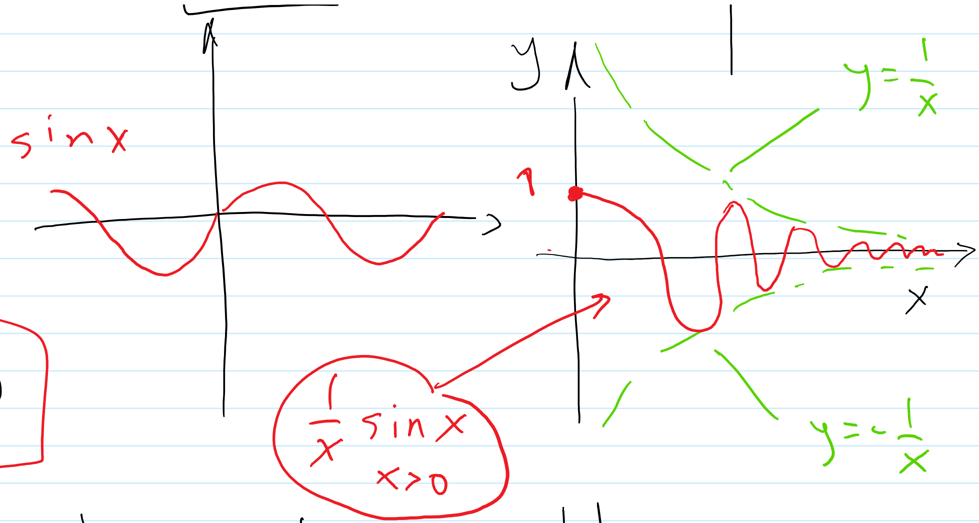
Example 3: $y = \dfrac{1}{x} \sin x$

$\boxed{x > 0}$

$$\left|\frac{1}{x} \sin x\right| \le \frac{1}{x}$$

$$\left|\frac{1}{x}\sin x\right| \le \frac{1}{x}$$

$$|\sin x| \le 1$$



$$\boxed{\lim_{x\to+\infty} \frac{1}{x}\sin x = 0}$$

$\Rightarrow$ $y=0$ is a horizontal asymptote for $y=\frac{1}{x}\sin x$

$$\boxed{-\frac{1}{x} \le \frac{1}{x}\sin x \le \frac{1}{x}}$$

? Does the function $\frac{1}{x}\sin x$ have a vertical asymptote at 0?

Recall: a function $g(x)$ has a vertical asymptote at $x=x_0$, if $\boxed{\lim_{x\to x_0^{\pm}} g(x) = \pm\infty}$

$$\lim_{x\to 0^+} \underbrace{\frac{1}{x}}_{\downarrow\ \infty}\underbrace{\sin x}_{\downarrow\ 0} = \lim_{x\to 0^+} \frac{\sin x}{x} = 1 \Rightarrow$$ $x=0$ is not a vertical asymptote.

$$\lim_{x\to 0^+} \frac{\cos x}{x} = \infty \qquad \lim_{x\to 0^-} \frac{\cos x}{x} = -\infty$$

$\lim_{x\to 0} \frac{\cos x}{x}$ does not exist

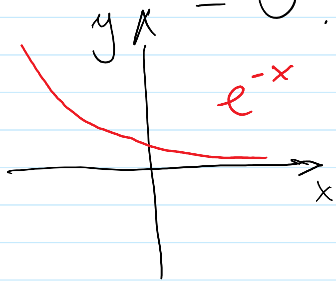
Example 4: $$\lim_{x\to\infty} \left(\underbrace{\sqrt{x^2+4}}_{\infty} - \underbrace{x}_{\infty}\right) =$$

Indeterminacy: $\boxed{\frac{0}{0}}$, $\boxed{\frac{\infty}{\infty}}$, $\boxed{\infty - \infty}$, $\boxed{0 \cdot \infty}$

$$= \lim_{x\to\infty} \frac{(\sqrt{x^2+4}-x)(\sqrt{x^2+4}+x)}{(\sqrt{x^2+4}+x)}$$

$$= \lim_{x\to\infty} \frac{\cancel{x^2}+4-\cancel{x^2}}{\sqrt{x^2+4}+x} = \lim_{x\to\infty} \frac{4}{\sqrt{x^2+4}+x} = 0.$$

Example 5: $\lim_{x\to\infty} e^{-x} = \lim_{x\to\infty} \frac{1}{e^x} = 0$



$$\lim_{x\to\infty} \frac{e^{3x} - e^{-3x}}{e^{3x}+e^{-3x}} = \lim_{x\to\infty} \frac{1-\frac{e^{-3x}}{e^{3x}}}{1+\frac{e^{-3x}}{e^{3x}}}$$

$e^{3x} \to \infty$, $e^{-3x} \to 0$

Model example: $\lim_{x\to\infty} \frac{x+1}{1-x} = \lim_{x\to\infty} \frac{1+\frac{1}{x}}{\frac{1}{x}-1} = -1.$

$\frac{1}{x} \to 0$

$$= \lim_{x\to\infty} \frac{1-e^{-6x}}{1+e^{-6x}} = 1.$$

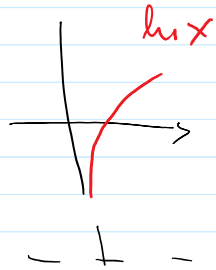
$e^{-6x} \to 0$

Example 6: $\lim_{x\to\infty} [x^2 - x] = \lim_{x\to\infty} x(x-1) = \infty$

$\infty \cdot \infty$

Example 7 $\lim_{x\to 0^+} \tan^{-1}(\ln x)$

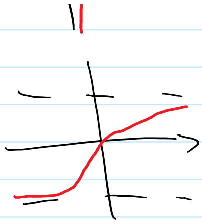
$\ln x \to -\infty$



$$= \left| \lim_{x\to 0^+} \ln x = -\infty \right|$$

$$= \left| \lim_{x \to 0^+} \ln x = -\infty \right| \qquad -\infty$$

$$= \lim_{t \to -\infty} \tan^{-1}(t) = -\frac{\pi}{2}$$



$$\text{So} \quad \lim_{x \to 0^+} \tan^{-1}(\ln x) = -\frac{\pi}{2}.$$