Homework 2.
Due October 7.

1. Show that if a function $f(z)$ satisfies the Cauchy-Riemann equations at a point z_0, then all directional derivatives of f at z_0 have the same value.

2. Let $P(z, \overline{z}) = \sum_{m,n} z^m \overline{z}^n$ be a polynomial in z and $\overline{z}$. Prove that if there exists at least one nonzero monomial for which $c_{m,n} \neq 0$ with $n > 0$, then the set of points where $P(z, \overline{z})$ is $\mathbb{C}$-differentiable is nowhere dense in $\mathbb{C}$.

3. Suppose that P is a polynomial and that $\frac{\partial}{\partial z} P = \frac{\partial}{\partial \overline{z}} P = 0$ for all z. Prove that $P \equiv \text{const}$.

4. Textbook, Ch. 2, Problem 4.

5. Textbook, Ch. 3, Problem 7.

6. (MATH 9024 only) Let $U = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. Let

$$\phi(z) = \frac{\alpha z + \beta}{\gamma z + \delta},$$

where α, β, γ, δ are real numbers with $\alpha\delta - \beta\gamma > 0$. Prove that $\phi : U \to U$ is one-to-one and onto.

Conversely, prove that if

$$\psi(z) = \frac{az + b}{cz + d}, \quad a, b, c, d \in \mathbb{C}$$

is one-to-one and onto as a function $\psi : U \to U$, then a, b, c, d are real (after multiplying the fraction by a scalar), and $ad - bc > 0$.