Sample Final.

This sample exam is different in length than the actual final exam. All problems are given for practice purposes only. Assume that closed contours are positively oriented.

1. For each of the following statements indicate whether the statement is true or false.

   (a) $e^{i\frac{\pi}{2}} = i$

   (b) The set $\{z \in \mathbb{C} : \ |Re(z)| < 1\}$ is simply connected.

   (c) $\log(1-i) = Log\sqrt{2} + i(-\frac{\pi}{4} + 2\pi k), k \in \mathbb{Z}$

   (d) If f is analytic in a domain D then the function $g(z) = \frac{df}{dz}$ is analytic in D.

   (e) If $f(z)$ is an entire function such that $f(0) = 1$ and $|f(z)| < 2$ for all $z \in \mathbb{C}$, then $f(z) = 1$ for all $z \in \mathbb{C}$.

   (f) (6) If $f(z)$ is an entire function and $f^{(5)}(2i) = 1$, then
   $$\int_{|z|=9} \frac{f(z)}{(z-2i)^6} dz = \frac{\pi i}{72}.$$

   (g)
   $$\sum_{k=0}^{+\infty} (\frac{1}{4i})^{2k} = \frac{16}{17}.$$

   (h) If $f(z)$ has a pole of order 5 at $z = 0$ and $g(z)$ has a zero of order 6 at $z = 0$, then the function $f(z)g(z)$ has a removable singularity at $z = 0$.

   (i) The function $f(z) = \frac{e^z}{z(z+1)^3}$ has a pole of order 3 at $z = -1$.

   (j) The function $f(z) = \cos(\frac{1}{z})$ has a simple pole at $z = 0$.

   (k) The function $f(z) = \frac{\sin z}{z}$ has a removable singularity at $z = 0$.

   (l) If a function $f(z)$ is analytic in a domain D and continuous up to the boundary, and $|f(z)|$ attains its maximum on the boundary of D, then f is constant.

   (m) If $Q(z)$ is a rational function which is also entire, then $Q(z)$ is constant.

2. Find all values of $(\frac{2i}{1-i})^{\frac{1}{5}}$.

3. Sketch the set
$$\{z \in \mathbb{C} \ : \ |z| \leq 2, -\frac{\pi}{2} \leq Arg(z) \leq 0\}.$$

4. Write the polynomial $z^2 + 3z - 1$ in the form of a Taylor series centred at $z_0 = i$.

5. Calculate the integral $\int_\Gamma (3z^2 + (\cos z)(\sin z))dz$, where Γ is the line segment from i to $i - 1$. Write the answer in the form $a + ib$, where a, b are real numbers.

6. Let $f(z) = \sum_{k=0}^{+\infty} \frac{k^5}{5^k} z^k$. Find $f''(0)$. Evaluate $\oint_{|z|=1} \frac{f(z)}{z^4} dz$.

7. Compute
$$\oint_{|z|=1} \frac{\cos z}{(z - \frac{\pi}{4})^{17}} dz.$$

8. Verify the Cauchy-Riemann equations for the function e^z.

9. Find the Laurent series for the following functions in the domain $|z| > 0$. Write your answer in the form $\sum_{k=-\infty}^{+\infty} a_k z^k$ and explain very clearly what the values of a_k are for each k.

   (a) $f(z) = \frac{\cos(2z)}{z^3}$

   (b) $g(z) = (z+1)e^z$.

   (c) $\sin \frac{1}{z}$.

10. Evaluate the integral
$$\oint_{|z|=2} \frac{e^{2iz}}{z(z-i)^2(z+3)} dz$$

11. Show that the function
$$f(x,y) = e^{x^2-y^2}(\cos(2xy) + i \sin(2xy))$$
is entire and find its derivative.

12. Solve the equation $e^{2z} + e^z + 1 = 0$.

13. How many root can a rational function have?

14. Prove that the function $f(z) = \overline{z}$ cannot be the uniform limit on the circle $|z| = 1$ of a sequence of holomorphic polynomials.

15. Prove that the function $f(z) = \int_0^1 \frac{\sin tz}{t} dt$ is entire

16. Find all automorphisms (conformal bijections) of the upper half-plane.