COMPLEX ANALYSIS II, MATH 9056, FALL 2017

HOMEWORK ASSIGNMENT 2.

Due Thursday, October 19.

2.1. Prove that any conformal self map of $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$ is fractional linear, i.e., $\mathrm{Aut}(\mathbb{C}^*)$ is the group of fractional linear (Möbius) transformations.

2.2. Define a *hyperbolic disc* centred at $z_0 \in \mathbb{D}$ of radius $r > 0$ as the set of all points $z \in \mathbb{D}$ such that $\rho(z, z_0) < r$. Show that any hyperbolic disc is a Euclidean disc, but the hyperbolic centre of the hyperbolic disc is not in general the Euclidean centre of the disc. (*Hint:* consider first a disc centred at the origin.)

2.3. Let $\{f_n\}$ be a sequence of holomorphic functions on a domain $\Omega \subset \mathbb{C}$ such that $\operatorname{Re} f_n(z) \geq 0$ for all $n \in \mathbb{N}$ and $z \in \Omega$. Prove that either there exists a subsequence f_{n_k} converging normally in Ω to a function f, or $f_n \to \infty$ uniformly on compacts in Ω.

2.4. Let $\{f_n\}$ be a locally bounded sequence of holomorphic functions on a domain $\Omega \subset \mathbb{C}$. Suppose that there exists a sequence of points $\{z_k\} \subset \Omega$ such that $z_k \to z_0 \in \Omega$, and that $\lim_{n\to\infty} f_n(z_k)$ exists for all k. Prove that $\{f_n\}$ converges normally on Ω.

2.5. Let $\Omega \subset \mathbb{C}$ be a domain. Recall that the infinitesimal form of the Carathéodory metric is given by $ds^2 = q_\Omega(z) dz d\bar{z}$, where

$$q_\Omega(z) = 4 \sup_{h \in \mathcal{O}(\Omega, \mathbb{D})} |h'(z)|^2.$$

Prove that q_Ω is continuous on Ω. This will show that the Carathéodory metric is a *Finsler* metric. (In general, the Carathéodory metric may not be a Riemannian metric because q_Ω may not be smooth.)