

## COMPLEX ANALYSIS II, MATH 9056, FALL 2017

### HOMEWORK ASSIGNMENT 4

Due November 23

- 4.1. Let  $\{f_n\}$  be a sequence of rational functions that converges normally to  $f(z)$  on the extended complex plane  $\mathbb{C}^*$ . Show that  $f_n(z)$  has the same degree as  $f(z)$  for large  $n$ . [Recall that  $\deg f$ , the *degree* of a rational function  $f(z) = \frac{p(z)}{q(z)}$ , is defined as  $\max\{\deg p, \deg q\}$ , where  $p(z)$  and  $q(z)$  are polynomials. It is known that if  $\deg f = d$ , then each value  $w \in \mathbb{C}$ ,  $w \neq f(\infty)$ , is assumed exactly  $d$  times (counting multiplicities).]
- 4.2. Show that the function
- $$e^{\frac{1}{z}} + e^{-\frac{1}{z}}$$
- omits only the value  $\infty$  at  $z = 0$ .
- 4.3. Let  $f(z)$  be analytic on the punctured disc  $\{0 < |z| < 1\}$  and define  $f_n(z) = f(z/n)$ ,  $n \geq 1$ . Show that  $\{f_n\}$  is a normal family on the punctured disc iff the singularity of  $f(z)$  at  $z = 0$  is removable or a pole.
- 4.4. Let  $D \subset \mathbb{C}$  be a bounded domain. Suppose  $f : D \rightarrow D$  is a holomorphic automorphism (conformal bijection). Let  $f_n(z) = f \circ f \circ \cdots \circ f$  ( $n$  times).
- (i) Prove that the sequence  $\{f_n\}$  has a subsequence that converges either to a constant or to an automorphism of  $D$ .
  - (ii) If the whole sequence  $\{f_n\}$  converges to  $g$ , then  $f(z) \equiv z$ .
- 4.5. Prove that any conformal bijective map between two rectangles in  $\mathbb{C}$  sending vertices to vertices is a linear map.