Homework 1.
Due September 23.

1. Let S be any set. Consider the space X consisting of bounded functions $f : S \to \mathbb{C}$. Let

$$||f|| = \sup\{|f(x)| : x \in S\}.$$

Prove that $(X, ||\cdot||)$ is a normed space. Under what condition on S is X a separable metric space (with the metric induced by the norm)? *(Recall that a space is separable if it contains a countable dense subset.)*

2. Given a vector space X (over $\mathbb{C}$), a metric d on X is called *translation-invariant* if for any $x, y, z \in X$,

$$d(x, y) = d(x - z, y - z). \quad (1)$$

(a) Prove that condition (1) is necessary for the metric d to be induced by a norm on X.

(b) Show that condition (1) is not sufficient.

(c) Show that if in addition to (1) the metric d also satisfies

$$d(\alpha x, 0) = |\alpha|\, d(x, 0), \text{ for all } x \in X, \alpha \in \mathbb{C}, \quad (2)$$

then the metric d comes from a norm on the space X.

3. Let $C[0, 1]$ be the space of continuous complex-valued functions on $[0, 1]$ equipped with the norm

$$||f|| = \sup\{|f(x)| : x \in [0, 1]\}.$$

Show that there does not exist an inner product $\langle \cdot, \cdot \rangle$ such that $\langle f, f \rangle = ||f||^2$. This will show that $C[0, 1]$ with this norm is not an inner product space.

4. Show that if $(X, \langle \cdot, \cdot \rangle)$ is an inner product space then the maps $x \mapsto \langle x, y \rangle$ and $x \mapsto \langle y, x \rangle$ are continuous. (For 9054 only: are these maps uniformly continuous? Explain.)

5. Let S be a subspace of a Hilbert space $\mathcal{H}$. Prove that $S^{\perp\perp} := (S^{\perp})^{\perp}$ is the closure of S.