Homework 2.
Due October 7.

1. Let $x_1, x_2, \ldots$ be a sequence of linearly independent vectors in an inner product space. Define vectors inductively by setting $e_1 = x_1/||x_1||$, and then for $n \geq 2$,

$$f_n = x_n - \sum_{j=1}^{n-1} \langle x_n, e_j \rangle e_j, \quad e_n = f_n/||f_n||.$$

Show that $\{e_n\}$ is an orthonormal sequence with the property that the linear span of $\{x_1, x_2, \ldots, x_n\}$ is the linear span of $\{e_1, e_2, \ldots, e_n\}$ for all n.

2. Let $\mathcal{H}$ be a Hilbert space. Prove the following:

(a) Every orthonormal set in $\mathcal{H}$ is linearly independent.

(b) If $\{e_1, e_2, \ldots, e_n\}$ is an orthonormal set in $\mathcal{H}$, and $M = \text{span}\{e_1, e_2, \ldots, e_n\}$, then M is closed, and the orthogonal projection $P : \mathcal{H} \to M$ is given by

$$Px = \sum_{i=1}^{n} \langle x, e_i \rangle e_i, \quad x \in \mathcal{H}.$$

3. Prove that a Hilbert space $\mathcal{H}$ admits a (at most) countable orthonormal basis iff $\mathcal{H}$ is separable.

4. A sequence $\{h_n\}$ in a Hilbert space $\mathcal{H}$ is said to *converge weakly* to $h \in \mathcal{H}$ if

$$\lim_{n\to\infty} \langle h_n, g \rangle = \langle h, g \rangle$$

for all $g \in \mathcal{H}$. Prove the following:

(a) If $\{e_n\}$ is an orthonormal sequence in $\mathcal{H}$, then $e_n \to 0$ weakly.

(b) If $h_n \to h$ in norm, then $h_n \to h$ weakly. Show that the converse is false, but if $h_n \to h$ weakly and $||h_n|| \to ||h||$ as $n \to \infty$, then $h_n \to h$ in norm.

5. Let V be a vector space and W be its linear subspace. For $x, y \in V$, define $x \sim y$ iff $x - y \in W$. Prove that this is an equivalence relation on V, and the set V/W of equivalence classes inherits the structure of a vector space.

(for Math 9054 only) Let V be a normed space with the norm $||\cdot||$, and W be a closed linear subspace. Define

$$||\xi|| = \inf_{x\in\xi} ||x||$$

for every $\xi \in V/W$. Prove that with this definition $(V/W, ||\cdot||)$ becomes a normed space. Discuss the dimension of V/W; prove that if V is a Banach space, then so is V/W.