Homework 3.

Due October 21.

1. Suppose $T : \mathcal{H} \to \mathcal{H}$ is a bounded linear operator on a Hilbert space $\mathcal{H}$ such that the range of T is one-dimensional. Show that there exist vectors x and y in $\mathcal{H}$ so that

$$Tz = \langle z, x \rangle y$$

for all $z \in \mathcal{H}$.

2. Let $C[0,1]$ be the Banach space of continuous complex-valued functions with the supremum norm. Let $\varphi : [0,1] \to [0,1]$ be a continuous function. Define the composition operator C_φ on $C[0,1]$ by $C_\varphi(f) = f \circ \varphi$. Prove that C_φ is a bounded linear operator and find its norm.

3. Let T be a diagonal operator in a Hilbert space $\mathcal{H}$ associated with a bounded sequence $\{\alpha_n\}$ of complex numbers (see Textbook, Example 2.8 on p. 33). Find the adjoint of T.

4. Let $L^2(0,1)$ be the space of square-integrable complex-valued functions on the interval $[0,1]$. Define the operator $M : L^2(0,1) \to L^2(0,1)$ by $M(f(x)) = xf(x)$ for $f \in L^2(0,1)$. Prove that M is a bounded linear operator and that the range of M is dense in $L(0,1)$ but is not closed.

5. Let $A \in \mathcal{B}(E)$, where E is a finite dimensional normed space. Prove that the following are equivalent

   (i) A is injective.

   (ii) A is surjective.

   (iii) A is invertible.

   (iv) There exists $B \in \mathcal{B}(E)$ such that $AB = \mathrm{Id}$.

   (v) There exists $B \in \mathcal{B}(E)$ such that $BA = \mathrm{Id}$.

   (For Math 9054 only) All statements above are inequivalent for infinite dimensional E. Illustrate that for any 5 different implications.