Homework 4.
Due November 18.

1. Use Baire's Category Theorem to prove that no infinite-dimensional Hilbert space can have a countable Hamel basis.

2. Show that if X is a Banach space that is not reflexive, then X^* is also not reflexive. (*Hint: Find a nonzero bounded linear functional on X^{**} which is 0 on $\{x^{**} : x \in X\}$.*)

3. Let ℓ^∞ be the space of bounded sequence equipped with the sup norm. Consider its subspace

$$F = \{\{a_n\} \in \ell^\infty : \exists N \in \mathbb{N} \text{ such that } a_n = 0, \forall n > N\},$$

where N depends on the sequence. Let $T_n : F \to \mathbb{C}$ be given by

$$T_m(\{a_n\}) = \sum_{n=1}^{m} a_n.$$

Show that each T_m is linear and bounded and for any fixed sequence $x = \{a_n\} \in F$, the quantity

$$\sup\{|T_m(x)| : m = 1, 2, 3, \dots\}$$

is finite. Is it true that $\sup\{||T|| : n = 1, 2, 3, \dots\}$ is finite? Does this contradict the Principle of Uniform Boundedness?

4. Prove that a linear operator $T : \mathcal{H} \to \mathcal{H}$ on a Hilbert space $\mathcal{H}$ that satisfies $\langle x, Ty\rangle = \langle Tx, y\rangle$ for all $x, y \in \mathcal{H}$ is always bounded.

5. Given a normed linear space X and a subspace $M \subset X$, define

$$M^\perp = \{\varphi \in X^* : \varphi(x) = 0 \text{ for all } x \in M\}.$$

This set is called the *annihilator* of M. Further, if $N \subset X^*$ is a subspace, define

$${}^\perp N = \{x \in X : \varphi(x) = 0 \text{ for all } \varphi \in N\},$$

i.e., ${}^\perp N$ is the set of common zeros of the bounded linear functionals in N. Show that for any subspace $M \subset X$, we have

$${}^\perp(M^\perp) = \overline{M},$$

where $\overline{M}$ is the closure of M in X.

* (For Math 9054 only) How does this notation correlate with the orthogonal complements in Hilbert spaces?