Homework 5.

Due December 2.

1. Suppose that $T \in \mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ and that T has closed range. Show that there exists $c > 0$ such that $||Th|| \geq c||h||$ for all $h \in (\ker T)^{\perp}$.

2. Let X be a Banach space and suppose T_1, T_2, S are bounded linear operators from X to X with T_1, T_2 compact. Show that $T_1 + T_2$, αT_1, ST_1 and $T_1 S$ are all compact ($\alpha \in \mathbb{C}$). If F is a finite rank operator, show that so is SF and FS.

3. Let $A \in \mathcal{B}(\mathcal{H})$ be a diagonal operator with the diagonal $\{\alpha_n\}$. Show that if A is compact, then $\lim_{n\to\infty} \alpha_n = 0$.

4. Show that every finite rank operator T in a Hilbert space $\mathcal{H}$ can be described as

$$Th = \sum_{j=1}^{n} \langle h, x_j \rangle y_j,$$

where x_j and y_j are some vectors, and $\{x_1, \ldots, x_n\}$ is an orthonormal set. (You may use the result of Problem 1 in Assignment 3).

5. (a) Let $T : \mathcal{H} \to \mathcal{H}$ be a compact operator on a Hilbert space $\mathcal{H}$. Prove that the closure of the range of T is a separable Hilbert space.

   (b) (For Math 9054 only) Show that if, in addition, T is self-adjoint and for some $k > 0$, $T^k = T \circ \cdots \circ T = 0$ (composition of T with itself k times), then $T \equiv 0$.