INTRODUCTION TO RIEMANN SURFACES

SHORT SUMMARY OF LECTURES

1. Basic definitions and examples

1. Definition of a holomorphic function $f : \Omega \to \mathbb{C}$, where Ω is a domain in $\mathbb{C}$. Unique continuation property of holomorphic functions.

2. An *analytic element* is a pair $F = (f, U)$, where $f \in \mathcal{O}(U)$ is a holomorphic function given by a convergent power series centred at a point $z_0 \in U$, and $U = \{z \in \mathbb{C} : |z - z_0| < R\}$ is the domain of convergence of the power series. We say that an analytic element $F = (f, U)$ is an *immediate analytic continuation* of an analytic element $G = (g, V)$, if $f|_{U\cap V} = g|_{U\cap V}$. We say that $F = (f, U)$ is an *analytic continuation* of $G = (g, V)$ if there exists a chain of analytic elements $H_j = (h_j, U_j)$, $j = 1, \ldots, n$, such that $F = H_1$, $G = H_n$, and H_j is an immediate analytic continuation of H_{j-1}. We say that F is an analytic continuation of G along a curve γ, if in addition the centres of U_j can be chosen to be on γ. By uniqueness, the analytic continuation does not depend on the choice of the analytic elements.

3. Monodromy Theorem: if an element F can be continued analytically along any path in a simply connected domain Ω, then F extends to a holomorphic function $F : \Omega \to \mathbb{C}$.

4. Definition: An analytic function is the union of all analytic elements which are obtained by analytic continuation of some analytic element F_0. Examples: $\sqrt{z}$, $\log z$, etc.

5. Two analytic functions agree if there exists a common analytic element. An analytic function can have at most countably many values at a given point (Poincaré - Volterra theorem).

6. Instead of analytic elements, one may consider *germs* of holomorphic functions.

7. A Riemann surface of an analytic function F on a domain $\Omega \subset \mathbb{C}$ can be defined as follows: Let $\mathcal{R}$ be the set of points $A = \{a, F_a(z)\}$, where $a \in \Omega$, and $F_a(z)$ is an analytic element defined on a disc centred at a. Given $\epsilon > 0$ and a point $A \in \mathcal{R}$, the ϵ-neighbourhood of A is defined to be the set

$$\{B = \{b, F_b(z)\} : |a - b| < \epsilon, F_b \text{ is an immediate analytic continuation of } F_a\}.$$

This defines a Hausdorff topology on $\mathcal{R}$. The projection $\pi : \mathcal{R} \to \Omega$ is given by $\pi(\{a, F_a(z)\}) = a$. Then π is a local homeomorphism. Using π we define the complex structure on $\mathcal{R}$, which turns $\mathcal{R}$ into a Riemann surface. It is called the Riemann surface of the function F. For example, the Riemann surface of $\sqrt{z}$ is a two-sheeted cover of $\mathbb{C} \setminus \{0\}$, and the Riemann surface of $\log z$ is an infinite-sheeted cover of $\mathbb{C} \setminus \{0\}$.

The Riemann surface of an analytic function F is the natural domain of existence of the holomorphic function $\pi \circ F$, which is single-valued on $\mathcal{R}$.

If we drop the assumption that $\mathcal{R}$ contains only analytic elements of one analytic function we will obtain a topological space containing analytic elements of all analytic functions. A Riemann surface of a particular analytic function is then a connected open set in $\mathcal{R}$. If we consider instead of analytic elements the germs of holomorphic functions, and define $\mathcal{R}$ similarly, we obtain the stale space of the sheaf of germs of holomorphic functions. (This will not be used in the course).

8. Definition: A complex manifold of complex dimension n is a topological manifold of dimension $2n$, whose transition function are biholomorphisms between open sets in $\mathbb{C}^n \cong \mathbb{R}^{2n}$. Theorem: A Riemann surface of an analytic function F is a complex manifold of dimension 1.

9. Definition: A Riemann surface is a complex manifold of dimension 1.

10: Examples of Riemann surfaces:

(1) domains in $\mathbb{C}$;

(2) $\mathbb{P}_1 \cong \mathbb{C} \cap \{\infty\}$. The smallest atlas required for this surface has two coordinate charts. It can also be given as a quotient of $\mathbb{C}^2 \setminus \{0\}$ under the equivalence relation $z \sim w <=> z = \lambda w$, $\lambda \in \mathbb{C} \setminus \{0\}$. $\mathbb{P}_1$ is compact.

(3) The quotient $\mathbb{C}/2\pi\mathbb{Z}$ is a Riemann surface, where $2\pi\mathbb{Z}$ is a subgroup of $\{\mathbb{C}, +\}$. It is conformally equivalent to $\mathbb{C} \setminus \{0\}$.

(4) Let ω_1, ω_2 be $\mathbb{R}$-linearly independent vectors in $\mathbb{C}$. Then

$$L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$$

is called a lattice. Again, it is a subgroup of $\{\mathbb{C}, +\}$. The quotient $X = \mathbb{C}/L$ is a compact Riemann surface. It is topologically $S^1 \times S^1$, and is called a *complex torus*.

(5) If $f : \Omega \to \mathbb{C}^n$ is a holomorphic map on a domain $\Omega \subset \mathbb{C}$, then

$$\Gamma_f = \{(z, w) : z \in \Omega, w = f(z)\},$$

the graph of f is a Riemann surface.

(6) A parametrized holomorphic curve in $\mathbb{C}^n$ is a holomorphic map $f : \Omega \to \mathbb{C}^n$. If the gradient $\nabla f \neq 0$, then its image $f(\Omega)$ is a Riemann surface. This follows from the holomorphic version of the implicit function theorem. The points where $\nabla f = 0$ are called the singular points of the curve. Near these points $f(\Omega)$ is not a complex manifold.

(7) If $f(z_1, z_2)$ is a holomorphic function on a domain $D \subset \mathbb{C}^2$, then the set

$$\{z = (z_1, z_2) \in \mathbb{C}^2 : f(z) = 0\}$$

is called a holomorphic curve. If $\nabla f(z) \neq 0$ for any z where $f(z) = 0$, then the holomorphic curve is a Riemann surface.

2. Algebraic Curves

1. An affine algebraic curve in $\mathbb{C}^2$ is a holomorphic curve defined by a holomorphic polynomial.

2. A projective space $\mathbb{P}_n$, $n \geq 1$, can be obtained as a quotient space

$$\left(\mathbb{C}^{n+1} \setminus \{0\}\right) / \sim, \text{ where } z \sim w <=> z = \lambda w, \; \lambda \in \mathbb{C} \setminus \{0\}.$$

The space $\mathbb{P}_n$ can be equipped with homogeneous coordinates $[z_0, \ldots, z_n]$. It is a compact complex manifold of dimension n. The set

$$U_0 = \{[z_0, \ldots, z_n] \in \mathbb{P}_n : z_0 \neq 0\}$$

can be identified with $\mathbb{C}^n$. The set $L_\infty = \{[z_0, \ldots, z_n] \in \mathbb{P}_n : z_0 = 0\} \cong \mathbb{P}_{n-1}$ is called the hypersurface at infinity.

3. A projective algebraic curve in $\mathbb{P}_2$ is defined as

$$\{[z_0, z_1, z_2] \in \mathbb{P}_2 : P(z_0, z_1, z_2) = 0\},$$

where $P(z)$ is a homogeneous polynomial in $\mathbb{C}^3$ of degree d, i.e., it satisfies $P(tz) = t^d P(t)$ for every $t \in \mathbb{C}$. A projective curve is a compact subset of $\mathbb{P}_2$.

4. Given an affine algebraic curve $X \subset \mathbb{C}^2$, its compactification in $\mathbb{P}_2$ can be defined as follows. If X is given by a polynomial $P(w_1, w_2)$ of degree $d > 0$ on $\mathbb{C}^2$, consider the projectivization of $P(w)$:

$$\hat{P}(z_0, z_1, z_2) = z_0^d P(z_1/z_0, z_2/z_0).$$

One can immediately verify that $\hat{P}(z)$ is a homogeneous polynomial of degree d. Its zero set in U_0 coincides with X, and thus the projective curve $\hat{P}(z) = 0\}$ is closure of X in $\mathbb{P}_2$.

5. The closure of a complex line in $\mathbb{C}^n$ is called a projective line.

6. Theorem: If $P(z_0, z_1, z_2)$ is a homogeneous polynomial of degree $d > 0$, and 0 is the only solution of

$$\frac{\partial P}{\partial z_0} = \frac{\partial P}{\partial z_1} = \frac{\partial P}{\partial z_2} = 0,$$

Then $X = \{P(z) = 0\} \subset \mathbb{P}_2$ is a compact Riemann surface.

3. Basic function theory on Riemann Surfaces

Holomorphic and meromorphic functions on Riemann surfaces. Local normal form of a holomorphic function: in a suitable coordinate system centred at a point z_0 any nonconstant holomorphic function F has the form $z \to z^m$, where m is a positive integer, and z_0 corresponds to the origin. The number m is called the multiplicity of the map F at the point z_0, notation: $m = \mathrm{Mult}_{z_0}(F)$. A meromorphic function on a Riemann surface can be viewed as holomorphic map into $\mathbb{P}_1$.

Theorem 3.1. *Let $F : X \to Y$ be a holomorphic map between compact connected Riemann surfaces. Then the quantity*

$$\mathrm{Deg}\,(F) = \sum_{z \in F^{-1}(w)} \mathrm{Mult}_z(F)$$

is independent of the choice of the point $w \in Y$. It is called the degree of the map F.

Corollary 3.1. *Any meromorphic function on $\mathbb{P}_1$ is a quotient of two homogeneous polynomials of the same degree.*

4. Classification of compact surfaces

Any orientable compact surface is homeomorphic to a sphere with g handles, $g \geq 0$. A sphere with g handles can be obtained by taking a connected sum of g tori. The nonorientable surface $\mathbb{R}P_2$ can be obtained by attaching a disc to the boundary of a Möbius band. Any compact nonorientable surface is homeomorphic to a connected sum of h copies of $\mathbb{R}P_2$.

Any Riemann surface is orientable, therefore, any compact Riemann surface is homeomorphic to a sphere with g handles. A compact Riemann surface admits triangulation, i.e., a decomposition of 2-simplices (images of a standard triangle in $\mathbb{R}^2$). The Euler characteristic of a surface S is $\chi(S) = F - E + V$, where F is the total number of faces (2-simplices), E is the number of edges (1-simplice), and V is the number of vertices (0-simplices) of a triangulation. The number $\chi(S)$ is independent of the choice of triangulation.

Theorem 4.1 (Riemann-Hurwitz Formula)**.** *Let $F : X \to Y$ be a nonconstant holomorphic map between compact connected Riemann surfaces. Then*

$$\chi(X) = \deg(F) \cdot \chi(Y) - \sum_{z \in X} \left(\mathrm{Mult}_z(F) - 1\right).$$