INTRODUCTION TO RIEMANN SURFACES.

1. Differential forms and topology of surfaces.

1.1. **Smooth manifolds.** Let M be a Hausdorff topological space. An n-dimensional chart on M is a pair (U, x), where U is an open subset of M and $x : U \to D$ is a homeomorphism onto an open subset $D \subset \mathbb{R}^n$. An n-dimensional topological space is simply M together with the collection of charts (U_α, x^α) covering M.

Let $x^\alpha = (x_1^\alpha, \ldots, x_n^\alpha)$ be local coordinates in (U_α), $x^\beta \circ (x^\alpha)^{-1} : x^\alpha(U_\alpha \cap U_\beta) \mapsto x^\beta(U_\alpha \cap U_\beta)$ are the transition functions. The manifold M is of class C^k, $k \leq \infty$, if all the transition functions are of class C^k. If we replace $\mathbb{R}^n$ by $\mathbb{C}^n \cong \mathbb{R}^{2n}$, and require the transition functions to be holomorphic, then M becomes a *complex manifold* of dimension n.

A function of class $C^k(M)$ is a map $f : M \to \mathbb{C}$ such that $f|_{U_\alpha} \circ (x_\alpha)^{-1} \in C^k(x^\alpha(U_\alpha))$ for all charts. A map $\phi : M \to N$ is of class C^k, if $\phi^* f = f \circ \phi \in C^k(M)$, for any $f \in C^k(N)$.

A collection $\{(U_\alpha, x^\alpha)\}$ that defines the structure of a C^k-manifold on M is called the *atlas* of M (of class C^k. If $k \geq 1$, then the Jacobian of $\frac{\partial x^\alpha}{\partial x^\beta}$ is well-defined on $U_a \cap U_\beta$. The atlas is oriented, if all the Jacobians are positive. With such choice of atlas the manifold M is called *oriented* (or *orientable* if such atlas exists but is not specified).

We will assume that all manifolds M are paracompact, i.e. $M = \sup_1^\infty K_j$, where K_j are compact.

Theorem 1.1. *(Whitney) Any paracompact n-dimensional manifold M of class C^k, $k \geq 1$, admits a C^k-embedding $\phi : M \hookrightarrow \mathbb{R}^{2n+1}$. Further, one can choose $\phi(M)$ to be a closed C^ω-submanifold of $\mathbb{R}^{2n+1}$.*

A manifold of class C^∞ will be called a *smooth* manifold.

1.2. **Vector fields.** A *vector field* v on a smooth manifold M is an operator of differentiation $v : C^k(M) \to C^{k-1}(M)$, which in a coordinate chart (U, x) has the form $v = \sum a_j \frac{\partial}{\partial x_j}$, $a_j \in C^{k-1}(U)$. It follows that in a different coordinate chart $(\tilde{U}, \tilde{x})$, we have

$$v = \sum \tilde{a}_\nu \frac{\partial}{\partial \tilde{x}_\nu} = \sum \left(\sum_j a_j \frac{\partial \tilde{x}_\nu}{\partial x_j} \right) \frac{\partial}{\partial \tilde{x}_\nu}.$$

This is the rule for change of coefficients of the vector field under the change of the coordinate chart.

Clearly that vector fields exist locally. Globally they can be constructed using the partition of unity, assuming that the atlas is locally finite. The latter can be always arranged by partitioning existing charts, and throwing away some of them. Partition of unity is a collection (λ_α), $\lambda_\alpha \in C^\infty(M)$, $\lambda_\alpha = 0$ outside U_α, $\lambda_\alpha \geq 0$, and $\sum_\alpha \lambda_\alpha \equiv 1$. If v_α are vector fields in coordinates charts, then $v = \sum \lambda_\alpha v_\alpha$ is a global vector field on all of M (we assume that v_α is zero outside U_α).

The tangent vector to M at a point $a \in M$ is the value of the vector field v at a point a, i.e. the functional $f \mapsto v(f)(a)$. $T_a M$ is a vector space of such functionals, which is called the tangent space t M at a. $T_a M$ is clearly spanned by functionals $\frac{\partial}{\partial x}|_a$, and therefore, it is a $\mathbb{R}$-linear space of dimension n. The set $TM = \bigsqcup T_\alpha M$ is called the *tangent bundle* of M. $TM|_{(U,x)} \cong U \times \mathbb{R}^n$ via

$$\sum c_j \frac{\partial}{\partial x_j}|_x \mapsto (x; c_1, \ldots, c_n).$$

These are, in fact, the coordinate charts on TM which it the structure of a smooth manifold with the natural projection $TM \to M$, $T_a M \to a$.

1.3. **Differential forms.** A differential of a function $f \in C^k(M)$, $k \geq 1$ is a linear operator on vector fields, acting as follows $(df)(v) = v(f) \in C^{k-1}(M)$ (if $v \in C^{k-1}(M)$). If locally $v = \sum c_j \frac{\partial}{\partial x_j}$, then $dx_\nu(\frac{\partial}{\partial x_j}) = \delta_{j\nu}$. Therefore,

$$v(f) = \sum c_j f'_{x_j} = \left(\sum f'_{x_\nu} dx_\nu\right)(v).$$

In particular, $df = \sum f'_{x_\nu} dx_\nu$.

A differential 1-form is defined as $\alpha = \sum a_j df_j$, $f_j \in C^k$, $a_j \in C^{k-1}$. By definition $\alpha(v) = \sum a_j v(f_j) \in C^{k-1}$. In a coordinate chart (U, x) the basis 1-forms are the differentials dx_j, and any form α can be written as $\alpha = \sum c_j dx_j$. Thus, under the change of coordinate charts, the change of coefficients of the form are given by

$$\alpha = \sum \left(\sum_j c_j \frac{\partial x_j}{\partial \tilde{x}_\nu}\right) d\tilde{x}_\nu.$$

Given two 1-forms α and β, the form $\alpha \wedge \beta$ is a bilinear antisymmetric operator on vector fields, acting on a pair of vector fields u and v as follows

$$(\alpha \wedge \beta)(u, v) = \det \begin{pmatrix} \alpha(u) & \alpha(v) \\ \beta(u) & \beta(v) \end{pmatrix}.$$

Note that $\beta \wedge \alpha = -\alpha \wedge \beta$, and $\alpha \wedge \alpha = 0$. Differential 2-form is, by definition, an operator of the form $\phi \sum c_{jl} \alpha_j \wedge \alpha_l$. In a local chart (U, x), $\phi = \sum_{j<l} c_{jl} dx_j \wedge dx_l$.

The operator d of the exterior differentiation is already defined above for functions $f \in C^k(M)$, $k \geq 1$. For 1-forms $\alpha = \sum c_j df_j$ we let $d\alpha = \sum dc_j \wedge df_j$. In particular, $d(df) = 0$. If $\dim M = 2$, then $d\phi = 0$ for all 2-forms.

In general one can define differential forms of arbitrary degree, however, on surfaces, forms of degree higher than two are zero, and we will not discuss them in this course. Smooth forms of degree $p \leq 2$ on M form a vector space, denoted by $\Lambda^p(M)$, in particular, $\Lambda^0(M) = C^\infty(M)$.

1.4. **Chains and integration.** A singular p-simplex on a smooth manifold M is the image of a standard p-dimensional simplex in $\mathbb{R}^p$ under a smooth map into M (together with its parametrization). A 0-dimensional simplex is a point on M, a 1-dimensional simplex is a smooth path, and a 2-dimensional simplex is a parametrized image of a triangle $\subset \mathbb{R}^2$.

Two (singular) p-simplexes are called *equal* if one can be obtained from another by a composition with some diffeomorphism.

A *p-chain* (singular chain of dimension p in M is a finite linear combination $\sigma = \sum n_j[T_j]$, where $[T_j]$ are singular p-simplexes in M, and $n_j \in \mathbb{Z}$. We call a chain *reduced* if all T_j are different. A chain is equal to zero if reduces to zero.

As an example consider an oriented boundary $[\partial T]$ of a standard simplex in $\mathbb{R}^p$. In the sequel we will deal with the chains of dimension at most two.

The *boundary* of a p-chain $\sum n_j[T_j]$ is a $p-1$-chain defined by

$$\partial\left(\sum n_j[T_j]\right) = \sum n_j[\partial T_j],$$

where ∂T_j is the image with respect to the map defining T_j of the boundary ∂T of the standard simplex in $\mathbb{R}^2$. For example,

(1) If $p = 0$, then $\partial(\sum n_j[a_j]) = 0$
(2) if $p = 1$, then $\partial(\sum n_j[\gamma_j]) = \sum n_j \partial[\gamma_j] = \sum n_j([b_j] - [a_j])$, where $a_j, b_j \in M$ are the starting and terminating points of γ_j.

A smooth 2-dimensional oriented surface (M, bM) with oriented boundary and with compact closure admits a finite triangulation, and therefore can be viewed as a 2-chain with coefficients 1 for all 2-simplexes of the triangulation, all the maps sending triangles into simplexes in $M \cup bM$ preserve the orientation.

A chain σ is called a *cycle* if $\partial\sigma = 0$, and *coboundary* if it has the form $\partial\tau$ for some chain τ. Cycles and coboundaries form a linear space over Z. It is easy to see that $\partial^2 = 0$, i.e., $\partial(\partial\sigma) = 0$ (this can be easily checked for a standard simplex in $\mathbb{R}^p$). In particular, every coboundary is also a cycle.

A p-th *cohomology group* of a manifold M, $H_p(M, \mathbb{Z})$, is a quotient group (p-cycles)/(p-coboundaries) with the identification $\sigma_1 \sim \sigma_2$ whenever $\sigma_1 - \sigma_2 = \partial\sigma$ for some $(p+1)$-chain σ. For example, if $p = 0$, then 0-coboundary is a chain of the form $\sum m_j([b_j] - [a_j])$, where a_j, b_j are the starting and the terminating points of the path γ_j on M. Let M_ν be the connected components of M. The 0-chain $\sum n_j[c_j]$ is the boundary if and only if $\sum_{c_j \in M_\nu} n_j = 0$ for any ν. Therefore, $H_0(M, \mathbb{Z}) \cong \mathbb{Z}^N$, if N is the total number of components (in general, it is $\bigoplus_\nu \mathbb{Z}$).

Define now the integral of a differential p-form on a q-chain. By definition, it is zero unless $p = q$. Further,

(1) Let $p = 0$, $\sigma = \sum n_j[a_j]$, and f is a function (0-form). Then

$$\int_\sigma f = \sum n_j f(a_j)$$

(2) Let $p = 1$, $\sigma = \sum n_j[\gamma_j]$, $\gamma_j : [0,1] \to M$, and $\alpha = \sum c_\nu df_\nu$. Then

$$\int_\sigma \alpha = \sum_{j,\nu} n_j \int_0^1 (c_\nu \circ \gamma_j) d(f_\nu \circ \gamma_j) = \sum_j n_j \int_0^1 (\gamma_j^* \alpha).$$

(3) Let $p = 2$, $\sigma = \sum n_j[T_j]$, $T_j : T \to M$, ϕ is a 2-form on M. Then

$$\int_\sigma \phi = \sum n_j \int_T (T_j^* \phi).$$

Theorem 1.2 (Stokes' Theorem)**.** *For any $\sigma, \phi \in C^1$,*

$$\int_{bM} \phi = \int_M d\psi.$$

In particular, $\int_{bM} \psi = \int_M d\psi$ for a compact oriented surface with compatibly oriented boundary.

1.5. **Poincaré Lemma.** We call a p-form α *closed* if $d\alpha = 0$, and *exact* if $\alpha = d\beta$.

Lemma 1.1. *In a star-shaped domain $D \subset \mathbb{R}^n$ any closed p-form, $p > 0$ is exact.*

Proof. We prove the lemma for $n = 2$, $p = 1$. Let $\alpha = adx + bdy$, $d\alpha = 0$. It follows that $(b'_x = a'_y$. Without loss of generality we may assume that $(0,0)$ be the centre of the star. Then $f(x,y) = \int_{(0,0)}^{(x,y)} \alpha$ (integration is along the straight segment. We have

$$\left(\int_{(0,0)}^{(x+\mathbb{D}x,y)} - \int_{(0,0)}^{(x,y)} \right) \alpha = \int_{\partial T} \alpha + \int +(x,y)^{(x+\mathbb{D}x,y)} \alpha = (a(x,y) + o(1)\mathbb{D}x,$$

where T is the triangle with the vertices $(0,0)$, (x,y), and $(x + \mathbb{D}x, dy)$. The integral along its boundary is zero by Stokes' theorem, and therefore, $\frac{\partial f}{\partial x} = a$. Analogously, $\frac{\partial f}{\partial y} = b$, and so $df = \alpha$. $\square$

Note that if α is bounded, and D is strictly star-shaped with smooth boundary, then $f \in C(\overline{D}) \cap C^1(D)$.

Lemma 1.2. *Let $D \subset \mathbb{C}$ be a domain, $a \in C^1(D)$. Then there exists $b \in C^1(D)$ such that $\frac{\partial b}{\partial \bar{z}} = a$.*

Proof. Recall the Cauchy-Green formula from Complex Analysis: if ϕ is a differentiable function with compact support in $\mathbb{C}$, then

$$\phi(z) = -\frac{1}{\pi}\int \frac{\partial \phi}{\partial \bar{\zeta}}\frac{dS_\zeta}{\zeta - z},$$

where $dS_\zeta = d\zeta \wedge d\eta = \frac{i}{2}d\zeta \wedge d\eta$, if $\zeta = \xi + i\eta$.

First consider the special case case when D and a are bounded. Let $b(z) = -\frac{1}{\pi}\int_D \frac{a(\zeta)dS_\zeta}{\zeta - z}d$. Since $a \in C^1(D)$, it is easily verified that b is continuous on $\overline{D}$, differentiable in D and holomorphic on $P_1 \setminus \overline{D}$. Further, for any $\phi \in C^1_c(D)$, we have

$$\int_D \frac{\partial b}{\partial \bar{z}}\phi dS_z = -\int_D b\frac{\partial \phi}{\partial \bar{z}}dS_z = \int_D a\left(\frac{1}{\pi}\int_D \frac{\partial \phi}{\partial \bar{z}}\frac{dS_\zeta}{\zeta - z}\right)dS_\zeta = \int_D \alpha\phi dS_\zeta.$$

Hence, $b'_{\bar{z}} = a$.

Now for the general case, use the representation $D = \cup_0^\infty D_\nu$, where $\bar{D}_\nu$ are compacts in $\mathbb{D}_{\nu+1}$, and $D \setminus D_\nu$ do not have compact connected components. Let

$$b_\nu(z) = -\frac{1}{\pi}\int_{D_\nu}\frac{a(\zeta)dS_\zeta}{\zeta - z}.$$

Then $\frac{\partial b_\nu}{\partial \bar{z}} = a$ in D_ν, and vanishes outside $\bar{D}_\nu$. It follows that $b_{\nu+1} - b_\nu = h_\nu \in \mathcal{O}(D_\nu)$. Therefore, by Runge's theorem there exist $\tilde{h}_\nu \in \mathcal{O}(D)$ such that $\left|\tilde{h}_\nu - h\nu\right| < 2^{-\nu}$ in $\bar{D}_{\nu-1}$. Let

$$b = b_1 + (h_1 - \tilde{h}_1) + \cdots + (h_\nu - \tilde{h}_\nu) + \cdots = b_\nu - \tilde{h}_1 - \cdots - \tilde{h}_{\nu-1} + (h_\nu - \tilde{h}_\nu) + \cdots.$$

Hence, $b \in C^1(D)$ and $b'_{\bar{z}} = a$ in D. □

Corollary 1.1. *In any domain $D \subset \mathbb{R}^2$ any smooth 2-form is exact.*

Proof. $\phi = adx \wedge dy = \frac{i}{2}adz \wedge d\bar{z}$. Then $a = b'_{\bar{z}}$, $db = b'_z dz + b'_{\bar{z}}d\bar{z}$. Therefore, $db \wedge dz = b'_{\bar{z}}d\bar{z} \wedge dz$, and so $\phi = d(\frac{1}{2i}bdz)$. □

1.6. **De Rham cohomology.** Since $d^2 = 0$, we may define on a smooth manifold M vector space (de Rham cohomology groups)

$$H^p_{DR}(M) = (\text{closed } p\text{-forms})/(\text{exact } p\text{-forms}),$$

the quotient space with respect to the equivalence relation $\phi_1 \sim \phi_2$ if $\phi_1 - \phi_2 = d\psi$.

The emphasize the coefficients we will denote by $H^p(M, \mathbb{C})$ and $H^p(M, \mathbb{R})$ the groups for compex-valued and real-valued forms respectively. This will be consistent with other known cohomology groups (singular, cellular, Čech, etc.)

For $p = 0$, the only exact form is the zero-form. Closed forms are locally constant forms. Therefore, $H^0(M, \mathbb{C}) = \mathbb{C}^N$, where N is the number of connected components of M. Analogously for $H^0(M, \mathbb{R})$. In particular, for a domain $D \subset \mathbb{R}^2$, $H^0(D, \mathbb{C}) = \mathbb{C}$. It follows from Lemma 1.1 that $H^2(D, \mathbb{C}) = 0$.

Lemma 1.3. *For a bounded domain $D \subset \mathbb{C}$ with $(m+1)$ connected components, $H^1(D, \mathbb{C}) = \mathbb{C}^m$.*

Proof. A closed 1-form α in D exact if and only if $\int_\gamma \alpha = 0$ for any closed path γ in D. The basis in $H_1(D, \mathbb{Z})$ is given by 1-cycles $[\gamma_j]$ around the "holes" of D, i.e., the compact components of $\mathbb{C} \setminus D$, such that $\mathbb{D}_{\gamma_j}\arg(z - a_\nu) = 2\pi\delta_{j\nu}$, where a_ν is a point in the ν-th hole. Then $\int_{\gamma_j} df = 0$, for any $f \in C^1(D)$. Therefore, for the equivalence class $[\alpha]$ of α in $H^1(M, \mathbb{C})$, the correspondence

$$[\alpha] \mapsto \left(\int_{\gamma_1}\alpha, \ldots, \int_{\gamma_m}\alpha\right) \in C^m,$$

is an embedding. On the other hand, for any $(c_1, \dots, c_m) \in \mathbb{C}^m$, the form

$$\alpha = \frac{1}{2\pi i}\left(\sum \frac{c_j}{z - a_j}\right) dz$$

is mapped by the above correspondence to $(c_1, \dots, c_m)$, and thus, this correspondence is an isomorphism. $\square$

1.7. **Surgery on an oriented surface.** Let M be a paracompact surface. The *exhaustion* function on M is a proper map $\rho : M \to \mathbb{R}$, i.e., the sets $\{\rho \le R\}$ are compact (or empty) for all $R \in \mathbb{R}$. If the surface is properly embedded into $\mathbb{R}^N$, which can always be done by Whitney's theorem, then the exhaustion function can be taken to be $|x|^2$. For $\rho \in C^\infty(M)$, by Sard's theorem for almost any R the set $\Gamma_R = \{\rho = R\}$ is a smooth curve and $d\rho \ne 0$ on Γ_R. By fixing a sequence $M_\nu = \{\rho \le R_\nu\} \to \infty$, we get an exhaustion of M by surfaces with boundary. M is a surface with boundary if it is embedded into a bigger surface $\widetilde{M}$ such that M is compact, and $\bar{M} \setminus M$ is a finite union of of pairwise disjoint simple smooth closed curves in $\widetilde{M}$. Assuming that $\widetilde{M}$ is orientable, these circles can be oriented in a compatible way with the orientation of M: if in a local coordinate chart $(U, (x, y))$ on $\widetilde{M}$ the surface M is given by $y > 0$, then the positive orientation on this part of the boundary is $\frac{\partial}{\partial x}$ (same as the orientation of R with respect to the upper half plane). The boundary oriented this way we will denote by bM. Thus, we have an oriented surface with boundary, that is a compatibly oriented pair (M, bM), where $M \cup bM$ is compact.

Near every component of the boundary there is a "collar", i.e. a neighbourhood V which is diffeomorphic to an annulus $K\{1 - \epsilon < |z| \le 1\}$ in the unit disc $\mathbb{D} \subset \mathbb{C}$. If M is given in $\widetilde{M}$ by the equation $\rho < 0$ with $d\rho \ne 0$ on bM, then for the collars can be constructed using the gradient lines of the function ρ as the preimages of radii and circles in $\mathbb{D}$. After that the "hole" corresponding to a component of the boundary can be "glued" by the constructed collar: by identifying first a point $x \in V$ with a point $z(x) \in K$, $x \sim z(x)$ and then factorizing by this identification, we get a smooth oriented surface $M' = (M??\mathbb{D})/\sim$ with fewer components of the boundary. Repeating this process for every hole, we get a simple compact surface $\widetilde{M}$, which contains our surface with boundary.

An important and non-trivial fact from differential geometry is that any smooth compact oriented surface is diffeomorphic to a sphere with a finite number of handles in $\mathbb{R}^3$. So any oriented surface with boundary is diffeomorphic to a sphere in $\mathbb{R}^3$ with g handles and m holes (with smooth boundaries).

The *genus* of the surface is the maximal number of simple closed pairwise disjoint curves γ_j on M such that $M \setminus \cup\gamma_j$ is connected. Therefore, in the realization of a surface in $\mathbb{R}^3$, the genus of a surface is g, i.e., the number of handles. The pair (g, m), where g is the genus and m is the number of components of the boundary, will be called the *type* of the surface M. In particular, it follows that any two surfaces of the same type are diffeomorphic.

For doing analysis on M it is sometimes convenient to "flatten" the surface. This can be done, for example, by cutting the handles: after removing the circles corresponding to the cuts, the remaining set is diffeomorphic to a domain in $\mathbb{C}$ with smooth boundary. Moreover, every cut in the surface corresponds to two connected components of the boundary of this domain, and the inverse map from the domain onto the surface extends smoothly to the closure of the domain with the non-vanishing differential on the boundary. Hence, the result of flattening is diffeomorphic to a circle with $2g - 1 + m$ holes.

The domain can be further cut further in $\mathbb{C}$ so the the resulting domain is simply-connected.

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Using ... one can verify that the Euler characteristic of a surface M of type (g, m) is equal to $\chi(M) = 2 - 2g - m$.

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