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CALC 1501 LECTURE NOTES

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5. Series

5.1. Basic Definitions. Given a sequence of real numbers

$$a_1, a_2, a_3, \dots, a_n, \dots$$

a formal expression

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots = \sum_{n=1}^{\infty} a_n \tag{5.1}$$

is called an *infinite series*, or just a *series*. We can add finitely many terms of the series to obtain

$$\begin{aligned} A_1 &= a_1, \\ A_2 &= a_1 + a_2, \\ A_3 &= a_1 + a_2 + a_3, \\ & \dots \\ A_n &= a_1 + a_2 + a_3 + \cdots + a_n, \\ & \dots \end{aligned}$$

The numbers $A_1, A_2, \dots, A_n, \dots$ are called the *partial sums* of the series (5.1). They naturally form a sequence $\{A_n\}$ of partial sums. If $A = \lim_{n\to\infty} A_n$, and A is a finite number, then the series $\sum a_n$ is called *convergent*, A is called its *sum*, and we write

$$A = \sum_{n=1}^{\infty} a_n.$$

If the sequence $\{A_n\}$ is divergent (i.e., A is infinite or does not exist), then the series (5.1) is also called *divergent*.

Example 5.1. Perhaps the simplest example of an infinite series is the so-called *geometric series*

$$a + aq + aq^2 + \cdots + aq^n + \cdots = \sum_{n=0}^{\infty} aq^{n-1}, \quad a \neq 0.$$

Its partial sum for $q \neq 1$ equals

$$A_n = \frac{a - aq^n}{1 - q}. \tag{5.2}$$

Indeed, by direct computation we obtain

$$A_n - qA_n = (a + aq + \cdots + aq^{n-1}) - q(a + aq + \cdots + aq^{n-1}) = a - aq^n,$$

from which equation (5.2) immediately follows.

By taking the limit in (5.2), we see that if $|q| < 1$ then the geometric series converges with the sum equal to $\frac{a}{1-q}$. If $|q| \geq 1$, then the series diverges. In particular, if $q = 1$, then $\lim A_n$ is either ∞ or $-\infty$, depending on the sign of a, and if $q = -1$, then the series takes the form

$$a - a + a - a + \dots,$$

and the value of partial sums alternates between a and 0. To summarize, the geometric series converges if $|q| < 1$, and

$$a + aq + aq^2 + \dots + aq^n + \dots = \sum_{n=0}^{\infty} aq^{n-1} = \frac{a}{1-q}.$$

◇

Example 5.2. Consider the series

$$\sum_{n=1}^{\infty} n \left(\frac{1}{2}\right)^n = \frac{1}{2} + \frac{1}{2} + \frac{3}{8} + \frac{1}{4} + \cdots.$$

This series resembles the geometric series, and we can try to find its sum using a similar technique. Let S_n be a partial sum of the first n terms. Then

$$S_n - \frac{1}{2}S_n = \left(\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \dots + \frac{n}{2^n}\right) - \left(\frac{1}{2^2} + \frac{2}{2^3} + \frac{3}{2^4} + \dots + \frac{n}{2^{n+1}}\right)$$

$$= \frac{1}{2} + \frac{2-1}{2^2} + \frac{3-2}{2^3} + \dots + \frac{n-(n-1)}{2^n} - \frac{n}{2^{n+1}} = \left(\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots + \frac{1}{2^n}\right) - \frac{n}{2^{n+1}}$$

The term in parentheses on the right-hand side of the above identity is the geometric series with $a = 1/2$ and $q = 1/2$, its partial sum was computed in the previous example:

$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots + \frac{1}{2^n} = \frac{1/2 - 1/2(1/2)^n}{1 - 1/2} = 1 - (1/2)^n.$$

From this we conclude that

$$S_n = \frac{1 - (1/2)^n - n/2^{n+1}}{1/2} = 2 - \frac{1}{2^{n-1}} - \frac{n}{2^n} \tag{5.3}$$

It follows that S_n converges to 2 as $n \to \infty$. Thus,

$$\sum_{n=1}^{\infty} n \left(\frac{1}{2}\right)^n = 2.$$

◇

Example 5.3. Determine the convergence of the series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}.$$

We estimate its partial sum:

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots + \frac{1}{\sqrt{n}} > n \cdot \frac{1}{\sqrt{n}} = \sqrt{n}.$$

We see that the partial sums grow indefinitely as n goes to infinity. Thus this series diverges. ◇

Let $\sum_{n=1}^{\infty} a_n$ be a series, and $A_m = \sum_{n=1}^{m} a_n$ be the partial sum. The quantity

$$R_m = \sum_{n=1}^{\infty} a_n - A_m = \sum_{n=m+1}^{\infty} a_n \tag{5.4}$$

is called the *remainder* of the series. We first observe that the series $\sum a_n$ converges if and only if the remainder R_m converges (as a series). Therefore, we may remove any finite (possibly very large!) number of elements from the series without affecting its convergence (or divergence). Further, if the series $\sum a_n$ converges, then by taking limit in (5.4) as $m \to \infty$ we see that $R_m \to 0$.

The next theorem gives a simple test to verify divergence of certain series.

Theorem 5.1. *If the series* $\sum_{n=1}^{\infty} a_n$ *converges, then* $\lim_{n\to\infty} a_n = 0$.

Proof. Let $A_n = a_1 + \cdots + a_n$. Then, since the series $\sum a_n$ converges, $\lim A_n$ exists as $n \to \infty$. Hence, $a_n = A_n - A_{n-1}$, and so $\lim_{n\to\infty} a_n = \lim_{n\to\infty}(A_n - A_{n-1}) = 0$. □

The *contrapositive* formulation of this theorem is sometimes called the *Test for Divergence*: if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum a_n$ diverges. For example, the series

$$\sum_{n=1}^{\infty} \frac{1}{1+s^n}$$

diverges for $0 < s \leq 1$ because $1/(1+s^n)$ does not converge to zero as $n \to \infty$. It is, however, wrong in general to conclude from the convergence of $\{a_n\}$ to zero that the series $\sum a_n$ converges. For instance, in Example 5.3 the series diverges, yet $\lim a_n = 0$.

Suppose now that the series $\sum a_n$ consists of positive terms. Then partial sums $\{A_n\}$ form an increasing sequence. If this sequence is bounded, then by the Monotone Convergence Theorem, it follows that the sequence of partial sums (and therefore the series) converges. On the other hand, if the sequence of partial sums is unbounded, then the series diverges. We illustrate this in the next example.

Consider the so-called *harmonic series*[1] given by

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n}. \tag{5.5}$$

Indeed, starting from the third term we can divide the series into groups consisting of $2, 4, 8, \ldots, 2^k, \ldots$ terms in each group:

$$\underbrace{\frac{1}{3} + \frac{1}{4}}_{2} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}_{4} + \underbrace{\frac{1}{9} + \cdots + \frac{1}{16}}_{8} + \ldots.$$

[1]The reason for the name is that every term of the series is the *harmonic mean* of the two neighbouring terms. Recall that the harmonic mean of two numbers a and b equals $\frac{2ab}{a+b}$. The harmonic mean is an important notion in geometry and physics.

Each group adds up to a number bigger than 1/2. Therefore, if we denote by H_n the partial sum of the first n terms of the series, we see that

$$\begin{aligned} &H_4 > 1/2 + 1/2 = 1 \\ &H_8 > H_4 + 1/2 > 1 + 1/2 = 3/2 \\ &H_{16} > H_8 + 1/2 > 3/2 + 1/2 = 2 \\ &\dots \\ &H_{2^k} > k \cdot 1/2 \\ &\dots \end{aligned} \tag{5.6}$$

Thus the sequence of partial sums is unbounded, and the harmonic series diverges. We note that as n grows, the value of the partial sum of n terms grows rather slowly. For example, Euler calculated that $H_{1000} \approx 7.48$ and $H_{1000000} = 14.39$.

Let us consider a more general series of the form

$$1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^s}, \tag{5.7}$$

where s is some positive real number. If $s = 1$, then (5.7) becomes the harmonic series. If $s < 1$, then the terms in (5.7) are bigger than the corresponding terms in (5.5), and so are the partial sums, hence, the series also diverges.

Now consider the case $s > 1$. We write $s = 1 + t$, where t is some positive number. We have

$$\frac{1}{(n+1)^s} + \frac{1}{(n+2)^s} + \cdots + \frac{1}{(2n)^s} < n \cdot \frac{1}{n^s} = \frac{1}{n^t}. \tag{5.8}$$

Splitting the series into groups, analogously to what we did for the harmonic series we have

$$\underbrace{\frac{1}{3^s} + \frac{1}{4^s}}_{2} + \underbrace{\frac{1}{5^s} + \frac{1}{6^s} + \frac{1}{7^s} + \frac{1}{8^s}}_{4} + \underbrace{\frac{1}{9^s} + \cdots + \frac{1}{16^s}}_{8} + \dots.$$

From (5.8) it follows that each group above is less than the corresponding term of the geometric series

$$\left\{\frac{1}{2^t}, \frac{1}{4^t}, \frac{1}{8^t}, \dots\right\} = \left\{\frac{1}{2^t}, \frac{1}{(2^t)^2}, \frac{1}{(2^t)^3}, \dots\right\}$$

Since this geometric series $\sum(\frac{1}{2^t})^n$ converges, we conclude that the sequence of partial sums of the series in (5.7) is bounded above, and therefore converges by the Monotone Convergence Theorem. Hence, the series (5.7) also converges. Another proof of convergence of the series for $s > 1$ will be given later, when we discuss the Integral Test for Convergence. The sums of this series are the values of a famous function $\zeta(s)$, called the Riemann ζ-function. It plays fundamental role in Number theory.

Example 5.4. Consider

$$\sum_{n=2}^{\infty} \frac{1}{n^2 - n}.$$

First observe that $\frac{1}{n^2-n} = \frac{1}{n-1} - \frac{1}{n}$. Therefore, the partial sum A_n of this series equals

$$\begin{aligned} A_n &= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) \\ &= 1 + \left(-\frac{1}{2} + \frac{1}{2}\right) + \cdots + \left(-\frac{1}{n-1} + \frac{1}{n-1}\right) - \frac{1}{n} = 1 - \frac{1}{n}. \end{aligned} \tag{5.9}$$

Thus $A_n \to 1$, and the series converges to 1. Series of this type are called *telescoping series.* ◇

5.2. **Comparison Theorems.** Convergence or divergence of series can be often determined by comparing a given series to another series, which is known to converge or diverge. In the next theorems we assume that $\sum_n^\infty a_n$, and $\sum_n^\infty b_n$ are series with positive terms

Theorem 5.2 (Comparison Test)**.** *Suppose that there exists a number $N > 0$ such that the inequality $a_n \leq b_n$ holds for all $n > N$. Then convergence of $\sum b_n$ implies convergence of $\sum a_n$. Equivalently, divergence of $\sum a_n$ implies that of $\sum b_n$.*

Proof. We may drop any finite number of terms of the series without affecting its convergence. Therefore, without loss of generality we may assume that that $a_n \leq b_n$ for all $n = 1, 2, \ldots$. Denote by A_n, and B_n the partial sums of $\sum a_n$ and $\sum b_n$ respectively. Then $A_n \leq B_n$. Suppose that $\sum b_n$ converges. Then the sequence of partial sums $\{B_n\}$ is bounded above: $B_n \leq L$, for some $L > 0$. Therefore $A_n \leq B_n \leq L$, and by the Monotone Convergence Theorem, the sequence $\{A_n\}$ also converges. This proves the theorem. □

Theorem 5.3 (Limit Comparison Test)**.** *Suppose there exists a limit*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = K \quad (0 \leq K \leq \infty).$$

Then:

(i) if the series $\sum b_n$ converges and $K < \infty$, then $\sum a_n$ converges.
(ii) if $\sum b_n$ diverges and $K > 0$, then $\sum a_n$ also diverges.

Proof. Suppose $\sum b_n$ converges with $K < \infty$. Given any $\varepsilon > 0$ by definition of the limit, for sufficiently large n we have

$$\frac{a_n}{b_n} < K + \varepsilon \implies a_n < (K + \varepsilon) \cdot b_n.$$

Since the series $\sum c_n = \sum (K+\varepsilon) b_n$ obtained by multiplying the series $\sum b_n$ by a constant $(K+\varepsilon)$ converges, we may apply the Comparison Test to $\sum a_n$ and $\sum c_n$ to conclude that the series $\sum a_n$ also converges.

The proof of the second statement is Exercise 5.9. □

Theorem 5.4. *Suppose there exists $N > 0$ such that for $n > N$ we have*

$$\frac{a_{n+1}}{a_n} \leq \frac{b_{n+1}}{b_n}. \tag{5.10}$$

Then convergence of $\sum b_n$ implies convergence of $\sum a_n$ and divergence of $\sum a_n$ implies that of $\sum b_n$.

The proof of this theorem is Exercise 5.12.

Example 5.5. Test for convergence the following series.

(i) $\displaystyle\sum_{n=1}^\infty \frac{(n!)^2}{(2n)!}$,

(ii) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\sqrt[n]{n}}$,

(iii) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n} - \ln\frac{n+1}{n}\right)$.

Solutions:

(i) $\displaystyle\frac{(n!)^2}{(2n)!} = \frac{1^2 2^2 3^2 \dots n^2}{1\cdot 2\cdot \dots \cdot n\cdot (n+1)\cdot \dots \cdot (2n)} = \frac{1}{n+1}\cdot\frac{2}{n+2}\cdot \dots \cdot \frac{n}{2n} < \frac{1}{2^n}$. Since $\sum \frac{1}{2^n}$ converges, it follows by the Comparison Test (Theorem 5.2) that the series $\sum \frac{n!}{(2n)!}$ also converges.

(ii) We use the Limit Comparison Theorem: Since

$$\frac{1}{n\sqrt[n]{n}} \div \frac{1}{n} = \frac{1}{\sqrt[n]{n}} \to 1,$$

and the harmonic series $\sum \frac{1}{n}$ diverges, we conclude that the series $\sum \frac{1}{n\sqrt[n]{n}}$ also diverges.

(iii) We use the inequality $\ln(1+x) \le x$, which holds for $-1 < x$ (See Lecture 1, Example. 1.12). First observer that

$$\ln\left(1+\frac{1}{n}\right) < \frac{1}{n} \quad \Longrightarrow \quad 0 < \frac{1}{n} - \ln\left(\frac{n+1}{n}\right).$$

Furthermore,

$$-\ln\frac{n+1}{n} = \ln\frac{n}{n+1} = \ln\left(1-\frac{1}{n+1}\right) < -\frac{1}{n+1}.$$

Therefore,

$$0 < \frac{1}{n} - \ln\frac{n+1}{n} < \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)} < \frac{1}{n^2}.$$

Thus, the series converges by the Comparison Test and convergence of $\sum \frac{1}{n^2}$. ⋄

5.3. **Alternating Series.** So far we have considered convergence of series with positive terms. The following result applies to *alternating series*, i.e., those in which signs of the general terms alternate between positive and negative.

Theorem 5.5 (The Alternating Series Test)**.** *Suppose that the sequence $\{b_n\}$ satisfies*

(i) $b_n \ge 0$ for sufficiently large n

(ii) b_n is a decreasing sequence starting from some large n.

(iii) $\lim_{n\to\infty} b_n = 0$.

Then the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n+1} b_n = b_1 - b_2 + b_3 - b_4 + \dots$$

converges.

Proof. First observe that even partial sums form an increasing sequence. Indeed, $S_{2n+2} = S_{2n} + (b_{2n+1} - b_{2n+2}) \ge S_{2n}$, $n \ge 1$. On the other hand, $S_{2n} \le b_1$. Therefore, by the Monotone Convergence Theorem, the sequence of even partial sums converges, say, $\lim_{n\to\infty} S_{2n} = S$. On the other hand, $S_{2n+1} = S_{2n} + b_{2n+1}$, and so

$$\lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty} S_{2n} + \lim_{n\to\infty} b_{2n+1} = S,$$

because $\lim_{n\to\infty} b_n = 0$. □

Example 5.6. Consider the series $\sum_{n=1}^{\infty}(-1)^n \sin\frac{1}{n}$. Since $\frac{1}{n} < \pi/2$, this is an alternating series with $b_n = \sin\frac{1}{n}$. Let $g(x) = \sin\frac{1}{x}$, then $g'(x) = -\frac{1}{x^2}\cos\frac{1}{x} < 0$ for $x > 1$. Therefore, $\{b_n\}$ is a decreasing sequence. Finally, $\lim_{n\to\infty}\sin\frac{1}{n} = 0$. Thus, by the Alternating Series Test, the series converges.

5.4. **Absolute and Conditional Convergence.** For series with arbitrary terms we may distinguished two types of convergence.

Definition 5.6. *A series $\sum_{k=1}^{\infty} a_k$ converges absolutely if the absolute value series $\sum_{k=1}^{\infty}|a_k|$ converges. If the series $\sum_{k=1}^{\infty} a_k$ converges but the series $\sum_{k=1}^{\infty}|a_k|$ diverges, we say that $\sum_{k=1}^{\infty} a_k$ converges conditionally.*

By putting absolute value we let all the terms of the series be positive which makes it more difficult for a series to converge, since negative terms can no longer cancel positive terms. So one may expect that certain series converge conditionally only. On the other hand, if a series of absolute values converges, then so does the original series, as the next theorem asserts.

Theorem 5.7. *If a series $\sum a_n$ is absolutely convergent, then it is convergent.*

Proof. We have $0 \le a_n + |a_n| \le 2|a_n|$. Therefore, by the Comparison Test, the series $\sum(a_n + |a_n|)$ converges. But this implies that the series $\sum a_n = \sum(a_n + |a_n|) - \sum|a_n|$ also converges. □

Example 5.7. The alternating harmonic series

$$\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

converges by the Alternating Series Test. The corresponding series of absolute values is the harmonic series (5.5) and so it diverges. Therefore, the series $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k}$ converges conditionally. ◇

Example 5.8. The series

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{(\sqrt{n}+1)(\sqrt[3]{n}+1)}$$

converges conditionally. This can be seen as follows: the series does not converge absolutely by the Limit Comparison Test (Thm 5.3) as compared to the divergent series $\sum_{n=1}^{\infty}\frac{1}{n^{5/6}}$. On the other hand, the original series converges by the Alternating Series Test. ◇

Example 5.9. The series

$$\sum_{k=1}^{\infty}\frac{\sin k}{2^k} = \frac{\sin 1}{2} + \frac{\sin 2}{2^2} + \frac{\sin 3}{2^3} + \cdots$$

has positive and negative terms, which appear in "unpredictable" order, and so the Alternating Series Test does not apply. To determine convergence we consider the absolute value series $\sum_{k=1}^{\infty}\frac{|\sin k|}{2^k}$. Since $|\sin k| \le 1$ for all k, we have

$$\frac{|\sin k|}{2^k} \le \frac{1}{2^k}.$$

The series $\sum_{k=1}^{\infty}\frac{1}{2^k}$ is a convergent geometric series. Therefore, the series $\sum_{k=1}^{\infty}\frac{|\sin k|}{2^k}$ converges by comparison. Thus, the original series $\sum_{k=1}^{\infty}\frac{\sin k}{2^k}$ converges absolutely, in particular, the series $\sum_{k=1}^{\infty}\frac{\sin k}{2^k}$ converges. ◇

5.5. **Integral, Ratio, and Root Tests for convergence.** In this subsection we consider some tests for convergence that can be used to determine convergence of certain series. The first test allows one to replace the question of convergence of a given series with the corresponding question concerning an improper integral.

Theorem 5.8 (Integral Test). *Let $f(x)$ be a continuous, positive and decreasing function on the interval $[A, \infty)$ for some $A \in \mathbb{R}$, and $f(n) = a_n$. Then,*
(i) if $\int_A^\infty dx$ converges, then so does $\sum_{n=1}^\infty a_n$;
(ii) if $\int_A^\infty dx$ diverges, then so does $\sum_{n=1}^\infty a_n$.

The proof of the test can be done by comparing the Riemann sums that are involved in the definition of the integral.

Example 5.10. The Integral Test is very effective in determining the convergence of the the so-called p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}, \quad p > 0.$$

The function $f(x) = \frac{1}{x^p}$ is continuous, positive, and decreasing for any $p > 0$. The improper integral $\int_1^\infty \frac{dx}{x^p}$ converges for $p > 1$ and diverges for $0 < p \leq 1$, therefore the p-series also converges only for $p > 1$. $\diamond$

Example 5.11. To determine whether the series $\sum_{n=2}^\infty \frac{1}{n \ln n}$ we first observe that the funtion $\frac{1}{x \ln x}$ is positive and decreasing for $x > 1$. By the Integral Test, to determine convergence of the series, we may consider the improper integral

$$\int_2^\infty \frac{dx}{x \ln x}.$$

We have

$$\int_2^\infty \frac{dx}{x \ln x} = \lim_{b \to \infty} \int_2^b \frac{dx}{x \ln x} = \lim_{b \to \infty} \ln(\ln x) \Big|_2^b = \infty,$$

where we used the substitution $y = \ln x$. Thus, by the Integral Test, the series $\sum_{n=2}^\infty \frac{1}{n \ln n}$ diverges. $\diamond$

The next two tests are based on the comparison of a given series with an appropriate geometric series.

Theorem 5.9 (Ratio Test). *Consider the series $\sum_{n=1}^\infty a_n$, with $a_n \neq 0$. Suppose that*

$$L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

exists. Then
(i) if $L < 1$ then the series converges absolutely;
(ii) if $L > 1$ then the series diverges;
(iii) if $L = 1$ or the limit does not to exist, then the test is inconclusive, i.e., there exist convergent and divergent series for which $L = 1$.

Proof. We will show that if $L < 1$ then the series converges. Let b be a number such that $L < b < 1$. Then there exists $N > 0$ such that for $n > N$ we have

$$\frac{|a_{n+1}|}{|a_n|} < b \;\Rightarrow\; |a_{n+1}| < b|a_n|.$$

Similarly, $|a_{n+2}| < b|a_{n+1}| < b^2 a_n$, ..., $|a_{n+k}| < b^k a_n$. Therefore, the series $\sum_{k=N}^{\infty} |a_k|$ satisfies $|a_k| < a_N b^k$. The geometric series $\sum_{k=1}^{\infty} a_N b^k$ converges, therefore, by the Comparison Test, the series $\sum |a_n|$ also converges.

The proof of the case $L > 1$ is Exercise 5.16. □

Example 5.12. Consider the series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{(-1)^n\, n^2}{2^n}.$$

To determine its convergence we use the Ratio Test. We have

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{(n+1)^2/2^{n+1}}{n^2/2^n} = \lim_{n\to\infty} \frac{(n+1)^2}{n^2} \cdot \frac{1}{2} = \frac{1}{2}.$$

Thus, the series converges absolutely. ⋄

Example 5.13. Consider the p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p},$$

where $p > 0$. Then the Ratio test is inconclusive for any value of p, but the Integral Test works. ⋄

Theorem 5.10 (Root Test)**.** *Given a series with positive terms* $\sum_{k=1}^{\infty} a_k$, *let*

$$L = \lim_{k\to\infty} \sqrt[k]{a_k}.$$

Then

(i) if $L < 1$ *, the series converges;*
(ii) if $L > 1$ *, the series diverges;*
(iii) if $L = 1$ *, the test is inconclusive.*

Proof. Suppose that $L < 1$. Let $\varepsilon > 0$ be so small that $L + \varepsilon < 1$. Then by the definition of the limit there exists $N > 0$ such that for all $n > N$ we have

$$L - \varepsilon \le \sqrt[k]{a_k} \le L + \varepsilon.$$

From this inequality we conclude that $a_k \le (L+\varepsilon)^k$. The geometric series $\sum_{k=1}^{\infty}(L+\varepsilon)^k$ converges, and therefore, by the Comparison Test, the series $\sum a_k$ also converges. □

Example 5.14. Test for convergence $\sum_{k=1}^{\infty} \frac{\sqrt{3^k}}{2^k}$. We have

$$\lim_{k\to\infty} \sqrt[k]{\frac{\sqrt{3^k}}{2^k}} = \lim_{k\to\infty} \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2} < 1.$$

Thus, the series converges, by the Root Test. ⋄

Exercises

5·1 Use the technique of Example 5.2 to find the values of q for which the series

$$\sum_{n=1}^{\infty} nq^n$$

converges.

5·2 Prove that if the series $\sum a_n$ converges then its remainder R_m as defined in (5.4) converges to zero.

5·3 Determine whether the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2+5n+6}$$

is convergent or divergent. If it is convergent, find its sum.

5·4 Find the value of c such that $\sum_{n=1}^{\infty} 2^{nc} = 2015$.

5·5 If the n-th partial sum of a series $\sum a_n$ is $S_n = 3 - n2^{-n}$, find a_n and $\sum_{n=1}^{\infty} a_n$.

5·6 Let $\sum_{n=1}^{\infty} a_n$ be a series with positive terms.

(a) Suppose that for any $n \geq 1$, the partial sum S_n satisfies $S_n < 100$. Prove that $\sum_{n=1}^{\infty} a_n$ converges.

(b) Suppose that for any $n \geq 1$,

$$a_n < \left(\frac{1}{2}\right)^n.$$

Prove that $\sum_{n=1}^{\infty} a_n$ converges.

In both parts, you do not need to find the sum of the series.

5·7 Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+s^n}$ converges or diverges for $s > 1$.

5·8 Find the sum of the series if it is converging: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+3)}$.

5·9 Prove part (ii) of Theorem 5.3.

5·10 Prove that the series in Example 5.6 converges conditionally only.

5·11 Test for convergence the following series:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n+2)}}$,

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$,

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(\ln n)^p}$, $p > 0$,

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(\ln n)^{\ln n}}$.

5·12 Prove Theorem 5.4. Hint: multiply equations (5.10) term by term, and use Theorem (5.2).

5·13 Show that if $a_n > 0$ and $\sum_{n=1}^{\infty} a_n$ is convergent, then $\sum_{n=1}^{\infty} \ln(1 + a_n)$ is also convergent.

5·14 Give an example that shows that it is possible for both $\sum a_n$ and $\sum b_n$ to diverge, but for $\sum a_n b_n$ to converge.

5·15 If $\sum a_n$ and $\sum b_n$ are both convergent series with positive terms, is it true that $\sum a_n b_n$ is also convergent? Justify your answer.

5·16 Prove the Ratio and the Root Test for the case $L > 1$.

5·17 Determine whether the series converges absolutely, conditionally, or diverges.

(i) $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\sqrt{n}}{n+10}$

(ii) $\sum_{n=1}^{\infty} n^5\left(\frac{2-3n}{4n+3}\right)^n$

(iii) $\sum_{n=1}^{\infty}\frac{\sin 5n}{n^5}$.

5·18 Let $\{f_n\}$ be the Fibonacci sequence, given by $f_1 = f_2 = 1$, $f_n = f_{n-1} + f_{n-2}$, for $n > 2$. Use Exercise 4.13 to assess the convergence of the series

$$\sum_{n=1}^{\infty}\frac{1}{f_n}$$

5·19 Determine the convergence of the series

$$\frac{2}{5} + \frac{2\cdot 6}{5\cdot 8} + \frac{2\cdot 6\cdot 10}{5\cdot 8\cdot 11} + \frac{2\cdot 6\cdot 10\cdot 14}{5\cdot 8\cdot 11\cdot 14} + \cdots$$

5·20 (i) Show that the following series converges.

$$\sum_{n=1}^{\infty}\frac{a^n}{n!}, \quad a > 0.$$

(ii) Explain how to use the result of part (i) to prove that $\lim_{n\to\infty}\frac{a^n}{n!} = 0$ for all $a > 0$.

5·21 Determine whether the series below converges absolutely, conditionally, or diverges.

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{3^n\, n!}{n^n}$$

Hint: Recall that $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$.

5·22 Assess the convergence of the following alternating series

$$\sum_{n=2}^{\infty}\left(\frac{1}{\sqrt{n}-1} - \frac{1}{\sqrt{n}+1}\right).$$