v: 2010-01-18

CALC 1501 LECTURE NOTES

RASUL SHAFIKOV

2. Factorization of Polynomials

2.1. **Complex Polynomials.** The set $\mathbb{R}$ of real numbers can be extended to a bigger set of the so-called *complex numbers*. This is done by introducing a single *imaginary* number $i = \sqrt{-1}$. Complex numbers can be written in the form $z = a + ib$, where $a, b \in \mathbb{R}$. In this representation a is called the *real* part of z, and b the *imaginary* part of z, denoted respectively by $\operatorname{Re} z$ and $\operatorname{Im} z$. Real numbers can be viewed as a subset of complex numbers with zero imaginary part. Thus, denoting the space of complex numbers by $\mathbb{C}$, we have the following chain of inclusions

$$\mathbb{N} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}.$$

We may extend the definition of arithmetic operations on real numbers to the space of complex numbers as follows:

(i) $(a + ib) + (a' + ib') = (a + a') + i(b + b')$

(ii) $(a + ib) \cdot (a' + ib') = (aa' - bb') + i(ab' + a'b)$

(iii) $\dfrac{a + ib}{a' + ib'} = \dfrac{aa' + bb'}{a'^2 + b'^2} + \dfrac{ba' - ab'}{a'^2 + b'^2} i, \quad \text{if} \quad a' + b'i \neq 0 = 0 + i0.$

One can verify that when $b = b' = 0$, the above formulas provide the usual operations of addition, multiplication and division for reals. Note that $i \cdot i = i^2 = -1$, which in particular means that the equation $z^2 + 1 = 0$ *over* the set of complex numbers has two complex roots: i and $-i$. This is in contrast with reals over which this equation has no solution.

With these operations on complex numbers we may define complex polynomials as functions on complex numbers defined by

$$P(z) = a_0 z^n + a_1 z^{n-1} + \dots + a_{n-1} z + a_n, \quad \text{where} \quad a_j \in \mathbb{C}. \tag{1}$$

If $a_0 \neq 0$, the n is called the degree of $P(z)$. In particular, if we ignore the choice of the letter for the unknown variable (x vs. z), the usual polynomials with real coefficients are examples of complex polynomials. (In other words, a polynomial is called *real* if in (**??**), $a_j \in \mathbb{R}$ for all j.) The following theorem is usually known as the *Fundamental Theorem of Algebra*.

Theorem 2.1. *Suppose $P(z)$ is a complex polynomial of degree $n > 0$. Then $P(z)$ has exactly n complex roots.*

In this theorem the number of roots should be counted with *multiplicity*, in other words, some roots may have to be counted more than once. For example, $z^2 + 2z + 1 = (z + 1)^2 = 0$ has two roots both of which are $z = -1$. In general, if $w_1, w_2, \dots, w_n$ (some of these can be the same) are the roots of a polynomial $P(z)$, then we can write

$$P(z) = a_0(z - w_1)(z - w_2)\dots(z - w_n). \tag{2}$$

This is called a *factorization* of a complex polynomial into complex linear factors. The proof of Theorem **??** requires some knowledge of complex analysis, a branch of mathematics that studies functions of complex variables.

Example 2.1. Consider the equation $f(z) = z^4 + z^2 = 0$. According to the Fundamental Theorem of Algebra, $f(z)$ has four roots. These can be easily found. Indeed, $z^4 + z^2 = z^2(z^2 + 1)$. Thus the roots are $z = 0$ (counted twice), $z = i$ and $z = -i$. So

$$f(z) = z^2(z - i)(z + i)$$

is a factorization of this polynomial into linear factors. ◇

Note that not every real polynomial admits a factorization into *real* linear factors (e.g., $x^2 + 1$).

2.2. **Factorization of Real Polynomials.** An important operation on complex numbers is *complex conjugation*, or just conjugation, which is denoted by a horizontal bar, and defined as follows:

$$\overline{a + ib} = a - ib.$$

In other words, to conjugate a complex number we simply change the sign of the imaginary part of the number. Note that if z is a real number, then $\overline{z} = z$, i.e., conjugation leaves real numbers unchanged. In addition to numbers, we can conjugate complex polynomials: if $P(z)$ is given as in equation (**??**), then

$$\overline{P(z)} = \overline{a_0}\,\overline{z}^n + \overline{a_1}\,\overline{z}^{n-1} + \cdots + \overline{a}_{n-1}\,\overline{z} + \overline{a}_n.$$

Here we used the fact that for $z, w \in \mathbb{C}$, we have $\overline{z \cdot w} = \overline{z} \cdot \overline{w}$, which can be verified directly.

Let $w = a + ib$ be a complex number. Then $\overline{w} = a - ib$. Consider the expression$(z - w)(z - \overline{w})$. Then

$$(z - w)(z - \overline{w}) = z^2 - wz - \overline{w}z + w\overline{w} = z^2 - (w + \overline{w})z + w\overline{w}. \tag{3}$$

We have $w + \overline{w} = (a + ib) + (a - ib) = 2a$, and $w\overline{w} = (a + ib)(a - ib) = a^2 + b^2$. Both are real numbers. Thus the product of two monomials as above with conjugate free terms yields a degree two polynomial with real coefficients.

Suppose now

$$P(z) = z^n + b_1 z^{n-1} + \cdots + b_{n-1}z + b_n, \quad \text{where} \ \ b_j \in \mathbb{R}, \tag{4}$$

is a polynomial of degree n with real coefficients. Let $w_1, w_2, \ldots, w_n$ be the complex roots of $P(z)$. We factorize $P(x)$ into complex linear factors as in (**??**), and conjugate. We get

$$\overline{P(z)} = (\overline{z} - \overline{w_1})(\overline{z} - \overline{w_2}) \ldots (\overline{z} - \overline{w_n}).$$

We introduce a new variable $\zeta = \overline{z}$. Then the above equation becomes

$$\overline{P}(\zeta) = (\zeta - \overline{w_1})(\zeta - \overline{w_2}) \ldots (\zeta - \overline{w_n}). \tag{5}$$

Since the coefficients of P are real, the left hand side in the above equation is the same polynomial as in (**??**), except that the unknown variable now is ζ instead of z. Therefore, the polynomial in (**??**) has the same collection of roots as $P(z)$. In fact, equation (**??**) tells us precisely what these roots are, namely, $\overline{w_1}, \overline{w_2}, \ldots, \overline{w_n}$, i.e., the roots of $\overline{P}(\zeta)$ are the conjugated roots of $P(z)$. But since $P(z)$ and $\overline{P}(\zeta)$ are the same polynomials, it follows that if w is a complex root of $P(z)$ and w is not a real number, then $\overline{w}$ is also a root of $P(z)$. Thus we may write

$$P(z) = (z - x_1)(z - x_2) \cdots (z - x_m)(z - z_1)(z - \overline{z_1}) \cdots (z - z_k)(z - \overline{z_k}), \tag{6}$$

where $x_1, \ldots, x_m$ are the real roots of $P(x)$, and $z_1, \overline{z_1}, \ldots, z_k, \overline{z_k}$ are the pairs of complex roots and their conjugates. Using the calculation in (**??**) we have

$$(z - z_1)(z - \overline{z_1}) = z - A_1 z + B_1, \quad \text{where} \ \ A_1 = 2\mathrm{Re}\, z_1, \ \text{and} \ B_1 = (\mathrm{Re}\, z_1)^2 + (\mathrm{Im}\, z_1)^2,$$

and similarly for the other pairs of complex conjugate roots of $P(z)$. Using this, and replacing z with x in (**??**) yields

$$P(x) = (x - x_1)(x - x_2)\cdots(x - x_m)(x^2 - A_1x + B_1)(x^2 - A_2x + B_2)\cdots(x^2 - A_kx + B_k),$$

where all the coefficients are real numbers. Thus, we proved the following theorem.

Theorem 2.2. *Suppose $P(x)$ is a real polynomial of degree $n > 0$. Then $P(x)$ admits factorization into a product of linear and quadratic factors with real coefficients.*

This theorem is used in the theory of integration of rational functions using partial fractions.

Example 2.2. Let $P(x) = x^4 + 1$. This polynomial does not have any real roots. Nevertheless, according to Theorem **??**, it can be factored into a product of two real polynomials. But what are these? One possible solution would be to find complex roots of $P(z)$ and then to multiply the conjugate monomials as discussed above. However, finding complex roots is not an easy task. Instead, we can try to factorize $P(z)$ into two polynomials $x^2 + ax + 1$ and $x^2 + bx + 1$ for some $a, b \in \mathbb{R}$. We get

$$(x^2 + ax + 1)(x^2 + bx + 1) = x^4 + ax^3 + x^2 + bx^3 + abx^2 + bx + x^2 + ax + 1.$$

We set this equal to $x^4 + 1$ and compare the coefficients of x^3, x^2, and x. It follows that $a = -b$ and $ab = -2$. So we may take $a = \sqrt{2}$ and $b = -\sqrt{2}$. This gives the required factorization:

$$x^4 + 1 = (x^2 + \sqrt{2}\,x + 1)(x^2 - \sqrt{2}\,x + 1).$$

◇