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CALC 1501 LECTURE NOTES

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3. The Gamma function

The Gamma function $\Gamma(x)$ is defined as an improper integral

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt. \tag{1}$$

This function brings together integration by parts and improper integrals. It can be seen as a solution to the following interpolation problem: find a smooth curve that connects the points (x, y) in the plane given by $y = 1 \cdot 2 \cdot 3 \cdot \dots \cdot x = x!$ at the positive integer values for x.

A plot of the first few factorials (see Fig 1.) makes clear that such a curve can be drawn, but it would be preferable to have a formula that precisely describes the curve, in which the number of operations does not depend on the size of n. The formula for the factorial $n!$ cannot be used directly for fractional values of n since it is only valid when n is a positive integer. There is, in fact, no such simple solution for factorials. Any combination of sums, products, powers, exponential functions or logarithms with a fixed number of terms will not suffice to express $n!$. But it is possible to find a general formula as an integral depending on a parameter. This was discovered by L. Euler in 1729. The symbol $\Gamma(x)$ and the name were proposed in 1814 by A.M. Legendre.

First consider the case $x = 1$. We have

$$\Gamma(1) = \int_0^\infty e^{-t} dt = \lim_{s\to\infty} \int_0^s e^{-t} dt = \lim_{s\to\infty} -e^{-t}\Big|_0^s = 1.$$

Further, using integration by parts, one can show that $\Gamma(n + 1) = n \cdot \Gamma(n)$. Indeed, for an integer $n \geq 1$,

$$\Gamma(n + 1) = \int_0^\infty t^{n+1-1} e^{-t} dt = \int_0^\infty t^n e^{-t} dt.$$

Consider the indefinite integral $\int t^n e^{-t} dt$. We apply integration by parts by choosing $u = t^n$, and $dv = e^{-t} dt$. Then $du = n\, t^{n-1} dt$ and $v = -e^{-t}$. According to the integration by parts formula, we have

$$\int t^n e^{-t} dt = -t^n e^{-t} - \int -e^{-t} n\, t^{n-1} dt = -t^n e^{-t} + n \int t^{n-1} e^{-t} dt.$$

Thus,

$$\int_0^\infty t^n e^{-t} dt = \lim_{s\to\infty} \left[-t^n e^{-t}\Big|_0^s + n \int_0^s t^{n-1} e^{-t} dt \right] \tag{2}$$

For the first term inside the limit above we get

$$\lim_{s\to\infty} \left(\frac{-t^n}{e^t}\right)\Big|_0^s = \lim_{s\to\infty} -\frac{s^n}{e^s}.$$

Using L'Hôpital's Rule n times we see that

$$\lim_{s\to\infty} \frac{s^n}{e^s} = \lim_{s\to\infty} \frac{-n!\, s^0}{e^s} = 0.$$

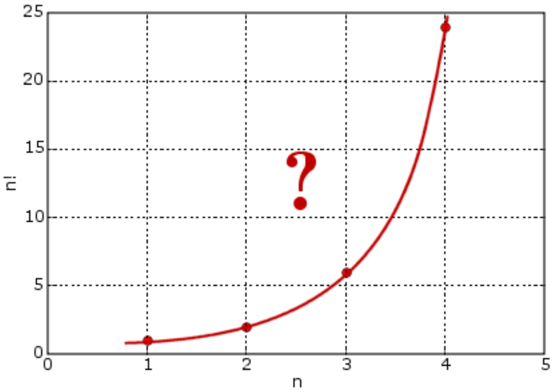

FIGURE 1. Interpolating $n!$

For the second term in (2) we have

$$\lim_{s\to\infty}\int_0^s t^{n-1}e^{-t}dt = \Gamma(n).$$

Combining everything together we have $\Gamma(n+1) = n\Gamma(n)$. This identity provides a reduction formula which can be used to compute inductively the values of the Gamma function for positive integers:

$$\Gamma(n+1) = n! \quad \text{where} \quad n \in \mathbb{N}.$$

Indeed, $\Gamma(2) = 1$; $\Gamma(3) = \Gamma(2+1) = 2\cdot\Gamma(2) = 2$; $\Gamma(4) = \Gamma(3+1) = 3\cdot\Gamma(3) = 3\cdot 2$, etc.

In fact, by inspection we see that our application of the integration by parts formula is valid not only for integer values n, but for all real $x > 0$ (see Exercises 3.1 and 3.2 for the case $0 < x < 1$), and so we have

$$\Gamma(x+1) = x\,\Gamma(x) \quad \text{for all} \quad x > 0. \tag{3}$$

Exercises

3.1. For $x \geq 1$ the above calculations show the convergences of the improper integral that defines the Gamma function. However, if $x < 1$, then the integral in (1) contains a negative power of t ($x-1$ becomes negative). Use the comparison test for improper integrals to show that the Gamma function is well-defined for $0 < x < 1$. (*Hint:* split the integral in (1) into two integrals.)

3.2. Verify formula (3) for the case when $0 < x < 1$.

3.3. Show that the integral in (1) diverges if $x \leq 0$.

3.4. The integral

$$\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$$

is called the *Gaussian* integral. It is particularly important in probability theory and statistics. Use the value of this integral to evaluate $\Gamma(1/2)$.

3.5. Use Problem 3.4 to calculate $\Gamma(5/3)$.

3.6. Prove that $\lim_{x\to 0^+}\Gamma(x) = +\infty$.