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CALC 1501 LECTURE NOTES

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4. Sequences

Definition 4.1. *A sequence s is a function $s : \mathbb{N} \to \mathbb{R}$. It can be thought of as a list of numbers*

$$s_1, s_2, s_3, \dots,$$

where $s_n = s(n)$ for $n \in \mathbb{N}$.

Example 4.1.

(i) $\{s_n\} = \left\{1, \frac{1}{2}, \frac{1}{3}, \dots\right\}$. The corresponding function $s : \mathbb{N} \to \mathbb{R}$ is given by $s(n) = \frac{1}{n}$.

(ii) Let

$$\{s_n\} = \left\{\frac{1}{2}, \frac{1}{6}, \frac{1}{12}, \frac{1}{20}, \dots\right\}.$$

Here $s(n) = \frac{1}{n\cdot(n+1)}$.

(iii) $\{s_n\} = \{1, -1, 1, -1, 1, -1 \dots\}$. For this sequence we can take, for example, $s_n = (-1)^{n+1}$.

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A sequence is defined *inductively* (or *recursively*) if $s_n = f(s_1, \dots s_{n-1})$, i.e., each term of the sequence is defined as a function of previously defined terms.

Example 4.2. (Fibonacci sequence[1].) By definition, the first two terms of the Fibonacci sequence $\{f_n\}$ are 1 and 1, and each consequent number is the sum of the previous two. Inductively this can be defined as follows.

$$f_1 = f_2 = 1, \quad f_n = f_{n-1} + f_{n-2}, \quad \text{for } n > 2.$$

The first several terms of the Fibonacci sequence can be easily computed to be

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \dots.$$

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Example 4.3. $s_1 = \sqrt{2}$, $s_n = \sqrt{2 + s_{n-1}}$ for $n > 1$. Then

$$s_1 = \sqrt{2}, s_2 = \sqrt{2+\sqrt{2}}, s_3 = \sqrt{2+\sqrt{2+\sqrt{2}}}, s_4 = \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}, \dots$$

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Definition 4.2. *A sequence $\{s_n\}$ converges to a real number L if for every positive number ϵ, there exists an $N \in \mathbb{N}$ such that whenever $n > N$ it follows that $|s_n - L| < \epsilon$. In this case we write*

$$\lim_{n\to\infty} s_n = L.$$

If $\{s_n\}$ does not converge, it is said to diverge.

[1] The Fibonacci sequence is named after Leonardo of Pisa, who was known as Fibonacci (a contraction of filius Bonaccio, "son of Bonaccio"). Fibonacci's 1202 book *Liber Abaci* introduced the sequence to Western European mathematics, although the sequence had been previously described in Indian mathematics.

The above definition is sometimes called the ϵ-N definition of convergence of a sequence.

Example 4.4. $\lim_{n\to\infty} \dfrac{1}{\sqrt{n}} = 0.$

To prove this, set $s_n = \frac{1}{\sqrt{n}}$, and $L = 0$. We need to show that given any $\epsilon > 0$, there exists an index $N > 0$ such that $|s_n - L| = |1/\sqrt{n}| < \epsilon$ for $n > N$. The inequality $1/\sqrt{n} < \epsilon$ is equivalent to $n > 1/\epsilon^2$. By taking $N = \lceil 1/\epsilon^2 \rceil$, we ensure that if $n > N$, then $|1/\sqrt{n}| < \epsilon$. (Recall that $\lceil x \rceil$ is the *ceiling* function; it equals the smallest integer bigger than or equal to x.) $\diamond$

Using a similar argument one can show that $\lim_{n\to\infty} \dfrac{1}{n^p} = 0$ for $p > 0$ (Exercise 4.1(i)).

Example 4.5. $\lim_{n\to\infty} \dfrac{n+1}{n} = 1.$

Let $s_n = \frac{n+1}{n}$, and $L = 1$. Then

$$|s_n - L| = \left| \frac{n+1}{n} - 1 \right| = \left| \frac{1}{n} \right| < \epsilon,$$

and therefore, the choice of $N = \lceil 1/\epsilon \rceil$ will ensure that $|s_n - L| < \epsilon$. $\diamond$

Example 4.6. $\lim_{n\to\infty} \left(\dfrac{1}{2}\right)^n = 0.$

Set $s_n = \left(\frac{1}{2}\right)^n$, $L = 0$. Then

$$\left| \left(\frac{1}{2}\right)^n \right| < \epsilon \iff n \ln(1/2) < \ln \epsilon \iff n > \frac{\ln \epsilon}{\ln(1/2)}.$$

Note that $\ln(1/2) < 0$, and so division by this number reverses the inequality. We could take $N = \lceil \frac{\ln \epsilon}{\ln(1/2)} \rceil$, but then N becomes negative for $\epsilon > 1$. So a better choice is

$$N = \max \left\{ \left\lceil \frac{\ln \epsilon}{\ln(1/2)} \right\rceil, 1 \right\}$$

$\diamond$

Definition 4.3. $\lim_{n\to\infty} s_n = \infty$ *means that for any real number M there exists an $N \in \mathbb{N}$ such that $s_n > M$ whenever $n \geq N$.*

Example 4.7. The Fibonacci sequence diverges to infinity. Indeed, starting with $n = 5$ we see that $f_n \geq n$. Therefore, given any number $M > 0$, $f_n > M$ for all $n > \lceil M \rceil$. $\diamond$

Example 4.8. Investigate convergence of $\{r^n\}$ for different values of $r > 0$.

Suppose $r > 1$. Then if $M > 0$ is arbitrary, the inequality $r^n > M$ is satisfied for $n > \dfrac{\ln M}{\ln r}$. Thus r^n diverges to infinity if $r > 1$. If $r = 1$, then r^n is a constant sequence 1, hence converges to 1. Finally, if $0 < r < 1$, then $\lim_{n\to\infty} r^n = 0$. Indeed, given $\epsilon > 0$, for $n > \max \left\{ \dfrac{\ln \epsilon}{\ln r}, 1 \right\}$ the inequality $r^n < \epsilon$ holds. $\diamond$

The following theorem provides a convenient way of calculating the limit by reducing the problem to algebraic manipulation of existing limits. It can be proved directly using the ϵ-N definition of convergence.

Theorem 4.4 (Algebraic Limit Theorem)**.** *If* $\lim a_n = A$, $\lim b_n = B$, *then*
(i) $\lim(ca_n) = cA$ *for* $c \in \mathbb{R}$,
(ii) $\lim(a_n + b_n) = A + B$,
(iii) $\lim(a_n \cdot b_n) = A \cdot B$,
(iv) $\lim\left(\frac{a_n}{b_n}\right) = \frac{A}{B}$, *if* $b_n \neq 0$ *and* $B \neq 0$.

Example 4.9. Examples of use of the Algebraic Limit Theorem.

1. $\lim_{n\to\infty} \frac{5n - 3n^2}{2n^2 + (-1)^n} = \lim_{n\to\infty} \frac{5/n - 3}{2 + \frac{(-1)^n}{n^2}} = -\frac{3}{2}$. Here we use the result of Exercise 4.1(i) and also the fact that $\lim_{n\to\infty} |a_n| = 0$ implies $\lim_{n\to\infty} a_n = 0$, which follows directly from the ϵ-N definition of convergence.
2. $\lim_{n\to\infty} \frac{n \ln n}{(n+1)^2} = \lim_{n\to\infty} \frac{n}{n+1} \cdot \lim_{n\to\infty} \frac{\ln n}{n+1} = 1 \cdot \lim_{n\to\infty} \frac{1/n}{1} = 0$. Here we used l'Hôpital's rule.

$\diamond$

A useful reduction for computing limits of sequences is the following: if f is a continuous function and $\{s_n\}$ is a sequence that converges to limit L, then $\lim_{n\to\infty} f(s_n) = f(L)$. Using this fact, one can prove that if a sequence is defined by an inductive formula

$$s_{n+1} = f(s_n), \tag{1}$$

where f is a continuous function, then assuming that the limit L of the sequence $\{s_n\}$ exists, it can be often found by taking the limit in (1): $L = f(L)$. Observe that $\lim s_n = \lim s_{n+1} = L$.

Example 4.10. Let $\{s_n\}$ be defined inductively by $s_1 = 1$, and

$$s_{n+1} = \frac{2s_n + 3}{4} \tag{2}$$

Assume that the limit of $\{s_n\}$ exists, say, $\lim_{n\to\infty} s_n = L$. Then we can take the limit as $n \to \infty$ on both sides of (2). We get

$$\lim_{n\to\infty} s_{n+1} = \lim_{n\to\infty} \frac{2s_n + 3}{4}.$$

it follows that

$$L = \frac{2L + 3}{4}, \quad \text{so} \quad L = 3/2.$$

Hence, $\lim s_n = 3/2$. $\diamond$

If *a priori* it is not known that the limit of $\{s_n\}$ exits, then the calculation of L from equation (1) may produce unpredictable results, see Exercise 4.4 for details. Thus, justification of existence of the limit becomes an important problem on its own.

Theorem 4.5 (Squeeze Theorem)**.** *If* $a_n \leq b_n \leq c_n$ *for* $n > n_0$ *and*

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L,$$

then $\lim_{n\to\infty} b_n = L$.

Proof. Take any $\epsilon > 0$. We need to find $N > 0$ such that $|b_n - L| < \epsilon$ whenever $n > N$. Since $a_n \to L$, there is $N_1 > 0$ such that for $n > N_1$ we have $|a_n - L| < \epsilon$. An equivalent form of this inequality is

$$L - \epsilon < a_n < L + \epsilon.$$

Similarly, since $c_n \to L$, there is $N_2 > 0$ such that $|c_n - L| < \epsilon$ for $n > N_2$, or

$$L - \epsilon < c_n < L + \epsilon.$$

Take $N = \max\{N_1, N_2\}$. Then for $n > N$ we have

$$L - \epsilon < a_n \le b_n \le c_n < L + \epsilon,$$

which implies that $|b_n - L| < \epsilon$. □

Example 4.11. $\lim_{n\to\infty} \frac{5^n}{n^n} = 0.$
To prove this we use the Squeeze Theorem. Indeed, for $n > 6$,

$$0 < \frac{5^n}{n^n} = \left(\frac{5}{n}\right)^n < \left(\frac{5}{6}\right)^n.$$

We may take $a_n = 0$, $b_n = \frac{5^n}{n^n}$, and $c_n = \left(\frac{5}{6}\right)^n$. Since $\lim_{n\to\infty} \left(\frac{5}{6}\right)^n = 0$ by Example 4.8, the Squeeze Theorem implies that $\lim_{n\to\infty} \frac{5^n}{n^n} = 0$. ◇

Definition 4.6. *A sequence $\{s_n\}$ is called increasing if $s_{n+1} \ge s_n$ for all n, strictly increasing if $s_{n+1} > s_n$ for all n. Decreasing and strictly decreasing sequences are defined similarly. Decreasing and increasing sequences are called* monotone *sequences.*

Example 4.12. $\left\{\frac{n}{n^2+1}\right\}$ is a decreasing sequence. This can be proved either by verifying the inequality $\frac{n+1}{(n+1)^2+1} \ge \frac{n}{n^2+1}$ for all n, or by showing that that function $f(x) = \frac{x}{x^2+1}$ has a negative derivative for $x > 1$. ◇

Definition 4.7. *An upper bound of a non-empty subset S of $\mathbb{R}$ is a number b such that $b \ge s$, for any $s \in S$. A number l is a least upper bound or supremum of S, denoted by* $\sup S$*, if l is an upper bound of S, and if b is another upper bound of S then $l \le b$.*

Example 4.13.

(1) $S_1 = \{0, 1/2, 2/3, 3/4, \dots\}$. Then $\sup S_1 = 1$.
(2) $S_2 = \mathbb{N}$. This set is unbounded, and therefore, the upper bound for this set does not exist.
(3) Let

$$S_3 = \{\sin n,\ n \in \mathbb{N}\} = \{\sin 1, \sin 2, \sin 3, \dots\}.$$

This set is bounded above by 1, since $\sin x \le 1$ for any x. But is there $\sup S_3$? If n could attain any real value, then since $\sin(\frac{\pi}{2} + 2\pi k) = 1$, the supremum would be 1. However, since $n \in \mathbb{N}$, $\sin n \ne 1$ for any n. Therefore, if $\sup S_3$ exists, in order to find it, one needs to investigate how close a natural number n can come to the set of numbers of the form $\frac{\pi}{2} + 2\pi k$, $k \in \mathbb{N}$.

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Lower bound and greatest lower bound (*infimum*) are defined similarly.

Definition 4.8. *A sequence $\{s_n\}$ is bounded above (below) if the set*

$$\{s_n; n \in \mathbb{N}\} = \{s_1, s_2, s_3, \dots\}$$

has an upper (lower) bound.

Axiom of Completeness. *Every nonempty set of real numbers that has an upper bound, has a least upper bound.*

The Axiom of Completeness distinguishes real numbers from rational numbers. For example, the set $S = \{x \in \mathbb{R} : x^2 < 2\}$ has a least upper bound $\sqrt{2}$. However, the set of rational numbers r, such that $r^2 < 2$, is bounded, but it does not have a least upper bound in $\mathbb{Q}$ ($\sqrt{2}$ is not rational!). Thus, the Axiom of completeness is false for rationals.

Let us return to Example 4.13(3). Since the set S_3 is bounded above by 1, the Axiom of Completeness guarantees that S_3 has a supremum, although it is a non-trivial problem to determine what it is.

Theorem 4.9 (Monotone Convergence Theorem)**.** *Every bounded monotone sequence converges.*

Proof. Consider the case when $\{s_n\}$ is an increasing sequence bounded above. Since the set $S = \{s_n;\ n \in \mathbb{N}\}$ is bounded, by the Axiom of Completeness, there exists $l = \sup S$. We claim that l is the limit of $\{s_n\}$. Indeed, take any $\epsilon > 0$. Then since l is the supremum of S, there exits an index N such that $s_N > l - \epsilon$. But since the sequence is increasing, we have $s_n > l - \epsilon$ for all $n > N$. This means that $|l - s_n| < \epsilon$ for $n > N$, which proves that $\lim s_n = l$.

The case when $\{s_n\}$ is decreasing and bounded below can be proved in a similar way. □

Example 4.14. Consider the sequence defined in Example 4.3. We may use induction to show that $s_n < 2$ for all n. Indeed, $s_1 = \sqrt{2} < 2$. If $s_n < 2$, then $2 + s_n < 4$. Taking the square root on both sides, we get $\sqrt{2 + s_n} < 2$, which means that $s_{n+1} < 2$. This shows that the inequality $s_n < 2$ holds for all n.

Further, $\{s_n\}$ is increasing. Indeed, $s_n < \sqrt{2 + s_n}$ is equivalent to $s_n^2 - s_n - 2 < 0$, which holds true for $-1 < s_n < 2$. By the previous paragraph $s_n < 2$, and therefore, $s_n < s_{n+1}$ for all n.

Thus, $\{s_n\}$ is a bounded monotone sequence, and by the Monotone Convergence Theorem $\{s_n\}$ converges. The limit L can be found by taking the limit as $n \to \infty$ on both sides of $s_n = \sqrt{s_n + 2}$. We have

$$L = \sqrt{2 + L} \implies L^2 - L - 2 = 0.$$

This equation has two roots: -1 and 2. Since $s_n > 0$ for all n, $L = 2$. ⋄

Exercises

4.1. Using only Definition 4.1 prove

(i) $\lim_{n\to\infty} \dfrac{1}{n^p} = 0,\ \ p > 0.$

(ii) $\lim_{n\to\infty} \dfrac{1 + 2n}{5 + 3n} = \dfrac{2}{3}.$

(iii) $\lim_{n\to\infty} \dfrac{\sin n}{n + 1} = 0.$

4.2. Give the definition of divergence of a sequence without referring to converge of a sequence. Use your definition to show that the sequence $s_n = (-1)^n + \frac{1}{n}$ diverges.

4.3. Give a definition of $\lim_{n\to\infty} s_n = -\infty$. Use your definition to verify that $\lim \log_a n = -\infty$ for $0 < a < 1$.

4.4. Let the sequence $\{s_n\}$ be defined inductively as $s_1 = 1$, and $s_{n+1} = {s_n}^2 - 1$ for $n > 1$. Compute L using the ideas of Example 4.10, and then show that this L cannot be the limit of the sequence s_n.

4.5. Use the Squeeze Theorem to find $\lim_{n\to\infty} \frac{\sin n + \cos n}{\sqrt{n}}$.

4.6. Prove that if a sequence $\{s_n\}$ converges, then the set $S = \{s_1, s_2, \dots\}$ is bounded.

4.7. Let $\{s_n\}$ be defined as $s_1 = 0.3$, $s_2 = 0.33$, $s_3 = 0.333$, ... Prove that $\{s_n\}$ converges.

4.8. Let $\{f_n\}$ be the Fibonacci sequence as defined in Example 4.2. Consider a sequence

$$s_1 = 1, \quad s_n = \frac{f_{n+1}}{f_n} \text{ for } n > 1.$$

Assume that s_n converges. Find its limit.

4.9. Show that the sequence $\{x_n\}$ defined by $x_1 = 3$, $x_{n+1} = \frac{1}{4-x_n}$ for $n > 1$, converges. Then find the limit.