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CALC 1501 LECTURE NOTES

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5. Taylor Series

Let $f(x)$ be a function that has derivatives of all orders on the interval $(a-R, a+R)$ for some $a \in \mathbb{R}$, and $R > 0$. Suppose that $f(x)$ can be represented on $(a-R, a+R)$ by a convergent power series

$$\sum_{n=0}^{\infty} c_n (x-a)^n. \tag{1}$$

This means that for any $x \in (a-R, a+R)$, the series (1) converges to $f(x)$. Then by direct differentiation of the power series (1), we see that $f^{(n)}(a) = n!\, c_n$, for all $n > 0$ (here $f^{(n)}$ denotes the derivative of $f(x)$ of order n). From this we conclude that

$$c_n = \frac{f^{(n)}(a)}{n!},$$

and thus the series in (1) becomes

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots \tag{2}$$

This is called the *Taylor series centred at $x = a$ associated with $f(x)$*. If $a = 0$, then (2) becomes

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots, \tag{3}$$

which is called the *Maclaurin series* associated with $f(x)$.

Example 5.1. Let $P(x)$ be a polynomial of degree N,

$$P(x) = c_0 + c_1 x + c_2 x^2 + \dots + c_N x^N.$$

By inspection, $c_n = \dfrac{P^{(n)}(0)}{n!}$ for $n = 1, \dots N$, and $c_n = 0$ for $n > N$. Thus, the Maclaurin series associated with $P(x)$ is exactly $P(x)$.

In general, however, one cannot immediately conclude that the Taylor or Maclaurin series associated with a function $f(x)$ converges to $f(x)$. In fact, it is not even clear whether the Taylor series of a given function converges at all. (Note that when we derived (2) we assumed to begin with that $f(x)$ has a power series representation.) Define the *Taylor polynomial* to be

$$T_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \dots + \frac{f^{(N)}(a)}{N!}(x-a)^N, \tag{4}$$

i.e., $T(x)$ is simply the order N partial sum of the Taylor series (2). Thus, by the definition of convergence, in order to show the convergence of the Taylor series to $f(x)$ we need to show that

$$\lim_{N \to \infty} T_N(x) = f(x) \tag{5}$$

for all x on some interval. If we define the *remainder* of the Taylor series to be

$$R_N(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n - T_N(x) = \frac{f^{(N+1)}(a)}{(N+1)!}(x-a)^{N+1} + \frac{f^{(N+2)}(a)}{(N+2)!}(x-a)^{N+2} + \dots, \tag{6}$$

then proving (5) is equivalent to showing

$$R_N(x) \to 0, \quad \text{as} \quad N \to \infty.$$

The following theorem provides a useful tool for proving convergence of Taylor series. For simplicity, we consider the case when $a = 0$. Then $T_N(x) = f(0) + f'(0)x + \dots \frac{f^{(N)}(0)}{N!}x^N$, and $R(x) = \frac{f^{(N+1)}(0)}{(N+1)!}x^{N+1} + \dots$.

Theorem 5.1 (Lagrange's Remainder Theorem)**.** *Let f be infinitely differentiable on $(-R, R)$. Then there exists a number c satisfying $|c| < |x|$ such that*

$$R_N(x) = \frac{f^{(N+1)}(c)}{(N+1)!}x^{N+1}. \tag{7}$$

Example 5.2. Let $f(x) = e^x$. Then $f^{(n)}(0) = e^0 = 1$ for all n. Therefore, $c_n = \frac{1}{n!}$, and we have

$$e^x \sim \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots.$$

The remainder of order N of this Maclaurin series is

$$R_N = \frac{x^{N+1}}{(N+1)!} + \frac{x^{N+2}}{(N+2)!} + \dots.$$

According to Lagrange's Remainder Theorem, there is a number c, $|c| < |x|$, such that

$$R_N(x) = \frac{f^{(N+1)}(c)}{(N+1)!}x^{N+1} = \frac{e^c}{(N+1)!}x^{N+1}.$$

For any fixed x, $R_N(x) \to 0$, since for any x, $\frac{x^n}{n!} \to 0$ as $n \to \infty$. Thus

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots.$$

for all $x \in \mathbb{R}$.

Example 5.3. Let

$$g(x) = \begin{cases} e^{-1/x^2}, & \text{if } \ x > 0 \\ 0, & \text{if } \ x \le 0 \end{cases} \tag{8}$$

Since e^{-1/x^2} approaches 0 as $x \to 0$, the function $g(x)$ is continuous at 0. In fact, using L'Hôpital's Rule one can show that $g(x)$ has continuous derivatives of any order at $x = 0$, and $g^{(n)}(0) = 0$ for any $n > 0$. The Maclaurin series associated to $g(x)$ is, therefore, identically zero. It follows that the Maclaurin series associated with $g(x)$ does not converge to $g(x)$ for $x > 0$.

Definition 5.2. *An infinitely differentiable function $f(x)$ is called real-analytic in a neighbourhood of a point $x = a$, if for some positive R the Taylor series* (2) *associated with $f(x)$ converges to $f(x)$ on $(a - R, a + R)$.*

Thus, e^x is a real-analytic function, while the function $g(x)$ in Example 5.3 is not real analytic near $x = 0$.

Proof of Lagrange's Remainder Theorem. . First note the following version of the Mean Value Theorem: If $f(x)$ and $g(x)$ are continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b) and $g'(x) \neq 0$, then there exists a point $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)} \tag{9}$$

This can be proved by applying the Mean Value Theorem to the function $h(x) = (f(b)-f(a))g(x) - (g(b) - g(a))f(x)$.

Note that the n-th order derivative of $R_N(x)$ at $x = 0$ vanishes for $n = 0, 1, 2, \ldots, N$. Therefore, if we apply (9) to functions $f(x) = R_N(x)$ and $g(x) = x^{N+1}$, then (assume $x > 0$ for simplicity) there exists a point $x_1 \in (0, x)$ such that

$$\frac{R_N(x)}{x^{N+1}} = \frac{R'_N(x_1)}{(N+1)x_1^N}.$$

We now repeat the process and apply (9) to functions $f(x) = R'_N(x)$ and $g(x) = x^N$ on the interval $(0, x_1)$: there is $x_2 \in (0, x_1)$ such that

$$\frac{R'_N(x_1)}{x_1^N} = \frac{R''_N(x_2)}{N x_2^{N-1}}.$$

Continue the process inductively N times. In the end we get

$$R_N(x) = \frac{x^{N+1}}{(N+1)!} \frac{R_N^{(N+1)}(x_{N+1})}{x_{N+1}^{N-N}},$$

where $x_{N+1} \in (0, x_N) \subset \cdots \subset (0, x)$. Now set $c = x_{N+1}$, then $c^{N-N} = 1$, and we can write

$$R_N(x) = \frac{R_N^{(N+1)}(c)}{(N+1)!} x^{N+1} = \frac{f^{(N+1)}(c)}{(N+1)!}\, x^{N+1},$$

where the last equality follows from the fact that $R_N^{(N+1)}(x) = (f(x) - T_N(x))^{(N+1)} = f^{(N+1)}(x)$, because $T_N^{(N+1)} \equiv 0$. This proves the theorem. □

Example 5.4. Let $f(x) = (1 + x)^{1/2}$. Then

$$f^{(n)}(0) = \frac{1}{2}\left(\frac{1}{2} - 1\right)\left(\frac{1}{2} - 2\right)\cdots\left(\frac{1}{2} - n + 1\right).$$

Therefore,

$$c_n = \binom{1/2}{n} = \frac{\frac{1}{2}\left(\frac{1}{2} - 1\right)\left(\frac{1}{2} - 2\right)\ldots\left(\frac{1}{2} - n + 1\right)}{n!},$$

and hence

$$(1 + x)^{1/2} \sim \sum_{n=0}^{\infty} \binom{1/2}{n} x^n$$

is the associated Maclaurin series. This is called the *binomial series.* Let us try use Lagrange's Remainder Theorem again to determine convergence of the series above. We have

$$R_N(x) = \frac{\frac{1}{2}\left(\frac{1}{2} - 1\right)\left(\frac{1}{2} - 2\right)\ldots\left(\frac{1}{2} - N\right)(1 + c)^{1/2-N}}{(N+1)!} x^{N+1}$$

for some c, $|c| < |x|$. If $|x| < 1$, then clearly $x^{N+1} \to 0$ as $N \to \infty$. Also, $\lim_{N\to\infty} \binom{1/2}{N} = 0$ (see Exercise 5.3). If $c > 0$, then we also have $(1+c)^{1/2-N} \to 0$ as $N \to \infty$. However, if $c < 0$, then $(1+c)^{1/2-N}$ does not go to zero, and we cannot be sure that $R_N(x)$ goes to zero.

In general, the binomial series converges for $x \in (-1, 1)$, and we have

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n, \quad k \in \mathbb{R}, \quad \text{and } |x| < 1.$$

Exercises

5.1. Show that the function $g(x)$ in Example 5.3 satisfies $g'(0) = 0$.

5.2. Use Lagrange's Remainder Theorem to prove that the Maclaurin series of $\cos x$ converges to $\cos x$ for all x.

5.3. Show that for any m,

$$\lim_{n\to\infty} \binom{m}{n} = 0.$$