CALC 1501 LECTURE NOTES

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These lecture notes are designed to provide supplementary material to Stewart, "Single Variable Calculus, Sixth Edition, with Early Transcendentals". More precisely, Section 1 is a complete replacement of **4.2** of Stewart. It is also a review of continuity and differentiability with the emphasis on rigourous definitions. This section also provides examples of proving inequalities using the Mean Value Theorem.

Section 2 of these notes offers some additional material regarding factorization of real polynomials. Although it is not required by the course curriculum, students are encouraged to read and understand this section at the time when integration using partial fractions is discussed.

Section 3 discusses the definition and basic properties of the Gamma function. It is a fine application of improper integrals and integration by parts. This section of the notes can be introduced after **7.8** in Stewart.

Section 4 is a replacement for **11.1** in Stewart. It emphasizes $\epsilon - N$ definition of convergence of sequences, introduces sup and inf of sets, and gives a proof of the Monotone Convergence and Squeeze Theorems.

Finally, Section 5 is a replacement for **11.10** of Stewart. Instead of Taylor's inequality given in Stewart without proof, Lagrange's Remainder Theorem is stated and proved. It is used then to prove that certain Taylor series converge to the corresponding functions. Real analytic functions are also defined.

After each section, except Section 2, a number of exercises are given. These can be used as part of homework assignments.

1. Mean Value Theorem

1.1. **Review: limit, continuity, differentiability.** We denote by $\mathbb{R}$ the set of real numbers. A *domain* D of $\mathbb{R}$ is any subset of $\mathbb{R}$. Typically this will be on *open* interval (a, b) or a *closed* interval $[a, b]$. A function of a real variable is a function $f : D \to \mathbb{R}$, where D is a domain of $\mathbb{R}$.

Definition 1.1 (The $\epsilon - \delta$ Definition)**.** *We say that a function $f(x)$ has a limit L as x approaches a point x_0 and write $\lim_{x\to x_0} f(x) = L$, if for any $\epsilon > 0$ there exists $\delta > 0$ such that whenever $0 < |x - x_0| < \delta$ (and $x \in D$) we have $|f(x) - L| < \epsilon$.*

The meaning of the above definition is that by choosing a sufficiently small interval $(x_0 - \delta, x_0 + \delta)$ of the point x_0 we can ensure that the values of $f(x)$ on this interval (excluding x_0) do not deviate from L by more than ϵ.

Definition 1.2. *We say that a function $f : D \to \mathbb{R}$ is continuous at a point $x_0 \in D$ if*

$$\lim_{x\to x_0} f(x) = f(x_0). \tag{1}$$

Using the $\epsilon - \delta$ definition this can be stated as follows: given $\epsilon > 0$, there exists $\delta > 0$ such that whenever $|x - x_0| < \delta$ we have $|f(x) - f(x_0)| < \epsilon$.

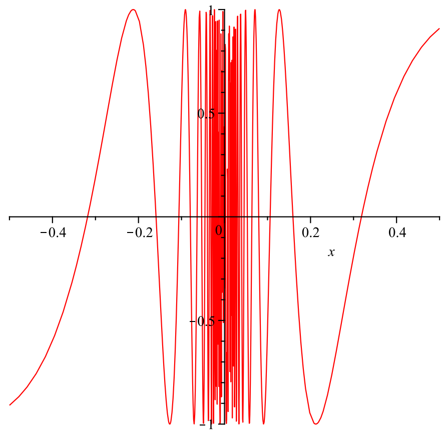

FIGURE 1. The graph of $\sin \frac{1}{x}$

Example 1.1. Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = x$. Let x_0 be any real number. Then $f(x)$ is continuous at x_0. Indeed, using the $\epsilon - \delta$ definition we have $|f(x) - f(x_0)| = |x - x_0| < \epsilon$. This inequality can be ensured by taking $\delta = \epsilon$. $\diamond$

Theorem 1.3. *If f and g are continuous functions on a domain D, then so are the functions $f+g$, $f \cdot g$, and $c \cdot f$, where c is any constant. The function f/g is continuous at all points of D where $g \neq 0$. Further, if g is a function defined on the range of f, then the function $g \circ f = g(f(x))$ is continuous on D.*

Using the above theorem and the fact that $f(x) = x$ is a continuous function as shown in Example 1.1, we conclude that any polynomial is a continuous function, and any rational function (the quotient of two polynomials) is continuous at all points where the denominator does not vanish.

Example 1.2. Let

$$f(x) = \begin{cases} 0, & x \neq 0 \\ 1, & x = 0 \end{cases}.$$

Then $\lim_{x\to 0} f(x)$ exists and equals zero, but it differs from the value of f at the origin since $f(0) = 1$. Therefore, equation (1) does not hold, and $f(x)$ is not continuous at the origin. However, letting $f(0) = 0$ will make this function continuous everywhere. $\diamond$

Example 1.3. Let

$$f(x) = \begin{cases} \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}.$$

The function f (see Fig. 1) is defined for all x. It is continuous for all $x \neq 0$ because it is a product of continuous functions x and $\sin 1/x$. But $f(x)$ does not have a limit as $x \to 0$ (why?), and therefore $f(x)$ is not continuous at the origin. Note that there is no choice of $f(0)$ that will make this function continuous at the origin. $\diamond$

Example 1.4. Let

$$f(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}.$$

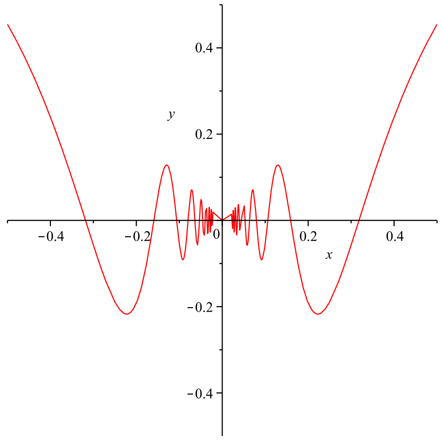

FIGURE 2. The graph of $x \sin \frac{1}{x}$

This function (see Fig. 2) is continuous everywhere. To prove the continuity at the origin, let us verify the $\epsilon - \delta$ definition. We have

$$|f(x) - f(0)| = \left| x \sin \frac{1}{x} \right| < \epsilon.$$

Since $|x \sin \frac{1}{x}| < |x|$ for all $x \neq 0$, we have

$$|f(x) - f(0)| = \left| x \sin \frac{1}{x} \right| < |x| < \epsilon,$$

and so we may take $\delta = \epsilon$. Intuitively, $\lim_{x \to 0} x \sin \dfrac{1}{x} = 0$ because $\sin \frac{1}{x}$ is bounded between -1 and 1, whereas x approaches zero. ◇

Definition 1.4. *Let $f(x)$ be defined on an interval $D \subset \mathbb{R}$. Let $x_0 \in D$. We say that $f(x)$ is differentiable at x_0 if the limit*

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \tag{2}$$

exists. The value of the limit is defined to be $f'(x_0)$, the derivative of f at x_0.

Example 1.5. Let

$$f(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}.$$

This function is continuous but not differentiable at the origin. The continuity was shown in Example 1.4. As for nondifferentiability, we have

$$\lim_{h \to 0} \frac{f(0 + h) - f(0)}{h} = \lim_{h \to 0} \frac{h \sin \frac{1}{h}}{h} = \lim_{h \to 0} \sin \frac{1}{h},$$

which does not exist. ◇

Example 1.6. Let

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}.$$

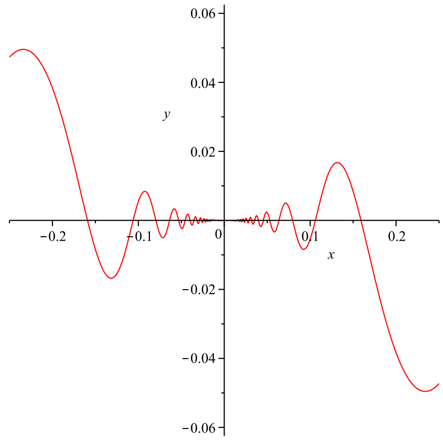

FIGURE 3. The graph of $x^2 \sin \frac{1}{x}$

This function (see Fig. 3) is continuous everywhere because it is the product of continuous functions x and $x \sin 1/x$ (as discussed in Example 1.4). To prove differentiability of this function at the origin let us compute the corresponding limit in (2).

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{h^2 \sin \frac{1}{h}}{h} = \lim_{h \to 0} h \sin \frac{1}{h}.$$

As we saw in Example 1.4 this limit equals 0. Thus $f'(0) = 0$. $\diamond$

1.2. **Mean Value Theorem.**

Definition 1.5. *Suppose $f(x)$ is a function defined on a domain D. The function $f(x)$ is said to have an absolute (global) maximum at a point $c \in D$, if $f(c) \geq f(x)$ for all $x \in D$. The number $f(c)$ is called the absolute (global) maximum value of f on the domain D. The function f has an absolute (global) minimum at $c \in D$, if $f(c) \leq f(x)$ for all $x \in D$. The number $f(c)$ is called the absolute (global) minimum value of f on the domain D.*

Theorem 1.6. *If $f(x)$ is continuous on a closed interval $[a, b]$, then $f(x)$ attains a maximum and a minimum value.*

The above theorem can be proved using the Axiom of Completeness for real numbers which will be stated when we discuss sequences.

Definition 1.7. *The function f defined on a domain D has a local maximum at a point $c \in D$, if there is an open interval $I \subset D$, such that $c \in I$, and $f(c) \geq f(x)$ for all $x \in I$. The function f has a local minimum at $c \in D$, if there is an open interval $I \subset D$, such that $c \in I$, and $f(c) \leq f(x)$ for all $x \in I$.*

Maxima and minima are called extreme points, or extrema.

Lemma 1.8. *Let $f(x)$ be a differentiable function on an interval (a, b). Suppose $x_0 \in (a, b)$. If $f'(x_0) > 0$, then for $x < x_0$ close to x_0 we have $f(x) < f(x_0)$, and $f(x) > f(x_0)$ for $x > x_0$ and close to x_0.*

The lemma above simply states that if $f'(x_0) > 0$, then $f(x)$ is an increasing function near x_0. A similar statement holds if we assume that $f'(x_0) < 0$ (see Exercise 1.2).

Proof. By definition,

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

If $f'(x_0) > 0$, then there exists a small interval $(x_0 - \delta, x_0 + \delta)$ such that

$$\frac{f(x) - f(x_0)}{x - x_0} > 0, \quad \text{for } x \neq x_0.$$

Suppose first that $x_0 < x < x_0 + \delta$. Then $x - x_0 > 0$, and from the above inequality we conclude that $f(x) - f(x_0) > 0$, or $f(x) > f(x_0)$. Now, if $x_0 - \delta < x < x_0$, then $x - x_0 < 0$, and the same inequality shows that $f(x) < f(x_0)$. □

Theorem 1.9 (Fermat's Theorem)**.** [1] *Let $f(x)$ be defined on an interval $[a, b]$, and suppose that $f(x)$ attains a maximal (or minimal) value at a point $c \in (a, b)$. If $f(x)$ is differentiable at $x = c$, then $f'(c) = 0$.*

Proof. We will assume that c is a maximum of $f(x)$, the case when c is a minimum can be treated in a similar way. Arguing by contradiction, suppose that $f'(c) \neq 0$. Then either $f'(c) > 0$ or $f'(c) < 0$. If $f'(c) > 0$, then Lemma 1.8 implies that $f(x) > f(c)$ for $x > c$ with x sufficiently close to c. Similarly, if $f'(c) < 0$, then $f(x) > f(c)$ for $x < c$. In both cases we see that $f(c)$ cannot be the maximum value of the function f. This contradiction proves the theorem. □

Geometrically, Fermat's theorem states that at extreme points the tangent line to the graph of the function f is horizontal, which should be intuitively clear. Also note, that if a maximal or a minimum value is attained at the end point of the interval $[a, b]$, then Fermat's theorem need not to hold.

Definition 1.10. *A point c is called a critical point of a differentiable function $f(x)$ if $f'(c) = 0$.*

Fermat's theorem now can be stated as follows: if c is a local maximum or minimum of a function $f(x)$, then c is a critical point of f. The converse to this statement is false: if $f'(c) = 0$, then it does not follow in general that c is a local maximum or a local minimum of $f(x)$. For example, if $f(x) = x^3$, then $f'(0) = 0$, but the origin is not an extreme point of x^3.

Theorem 1.11 (Rolle's Theorem)**.** [2] *Suppose $f(x)$ is continuous on the interval $[a, b]$ and differentiable on (a, b), and $f(a) = f(b)$. Then there exists a number $c \in (a, b)$ such that $f'(c) = 0$.*

Proof. By Theorem 1.6, a continuous function on a closed interval $[a, b]$ attains its maximum value, say, M, and its minimum value, say, m. Consider two cases:

1. Suppose $M = m$. Then $f(x)$ on $[a, b]$ is a constant function, since $m \leq f(x) \leq M = m$ for all $x \in [a, b]$. Therefore, $f'(x) = 0$ for all x.

2. Suppose $M > m$. Since $f(a) = f(b)$, we know that either M or m is attained at some point c inside the interval (a, b), (i.e., not at the end points of the interval). In this case, it follows from Fermat's theorem that $f'(c)$ must be zero. □

Geometrically, Rolle's theorem states that if $f(a) = f(b)$, then there is a point c between a and b such that the tangent line to the graph of f at point c is horizontal. This occurs at a local maximum or a local minimum of $f(x)$.

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[1] This is a modern formulation of the theorem. It captures the essence of Fermat's method for finding maximal and minimal values of a function. The notion of derivative was not yet developed at Fermat's time.

[2] Despite the name, Rolle only suggested this result for polynomials.

Theorem 1.12 (Mean Value Theorem)**.** *Suppose that $f(x)$ is continuous on $[a,b]$ and differentiable on (a,b). Then there exists a point $c \in (a,b)$ such that*

$$\frac{f(b)-f(a)}{b-a} = f'(c).$$

Proof. Define an auxiliary function

$$F(x) = f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x-a).$$

This function satisfies the conditions of Rolle's theorem. Indeed, it is continuous on $[a,b]$, because it is a difference of a continuous function $f(x)$ and a linear (hence continuous!) function

$$f(a) - \frac{f(b)-f(a)}{b-a}(x-a).$$

On the interval (a,b), we have

$$F'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}.$$

Finally, $F(a) = f(a) - f(a) = 0$, and $F(b) = f(b) - f(a) - \frac{f(b)-f(a)}{b-a}(b-a) = f(b) - f(a) - (f(b) - f(a)) = 0$, and so $F(a) = F(b)$.

Therefore, we may apply Rolle's theorem to the function $F(x)$, and so there exists a point $c \in (a,b)$ such that $F'(c) = 0$. This means that

$$f'(c) - \frac{f(b)-f(a)}{b-a} = 0.$$

This implies

$$f'(c) = \frac{f(b)-f(a)}{b-a},$$

which is exactly what we wanted to prove. □

1.3. **Proving inequalities.** The Mean Value Theorem can be used for proving inequalities.

Example 1.7. Prove that if $x > 0$, then

$$\ln(1+x) < x.$$

Solution. Let $a = 0$, $b = x$, and $f(x) = \ln(1+x) - x$. Then $f'(x) = \frac{1}{1+x} - 1 = -\frac{x}{1+x}$. By the Mean Value Theorem applied to the function f on the interval $[a,b] = [0,x]$, there exists a point $c \in (0,x)$ such that

$$f'(c) = \frac{f(x)-f(0)}{x-0},$$

or

$$-\frac{c}{1+c} = \frac{\ln(1+x)-x}{x}. \tag{3}$$

Note that $c > 0$, and therefore, $-\frac{c}{1+c} < 0$. Therefore, equation (3) implies

$$\frac{\ln(1+x)-x}{x} < 0.$$

Since $x > 0$, the numerator in the above inequality must be negative, i.e.,

$$\ln(1+x) - x < 0,$$

which is what we had to prove. ◇

Example 1.8. Prove that if $x > 0$, and $n > 1$, then

$$(1 + x)^n > 1 + nx.$$

Solution. Let $a = 0$, and $b = x$, and $f(x) = (1 + x)^n - (1 + nx)$. Then $f'(x) = n(1 + x)^{n-1} - n$, and by the Mean Value Theorem, we have

$$n(1 + c)^{n-1} - n = \frac{(1 + x)^n - (1 + nx) - 0}{x} \tag{4}$$

for some $c \in (0, x)$. Note that $1 + c > 1$, and for $n > 1$, we have $(1 + c)^{n-1} > 1$. Therefore,

$$n(1 + c)^{n-1} - n > 0.$$

From this and equation (4) we conclude that

$$\frac{(1 + x)^n - (1 + nx)}{x} > 0.$$

Since $x > 0$, this yields the desired inequality. ⋄

Exercises

1.1. Show that the function in Example 1.6 does not have the second order derivative at $x = 0$.

1.2. Formulate and prove a statement similar to Lemma 1.8 for the case when $f'(x_0) < 0$.

1.3. Give an example of a function which is defined on the closed interval $[0, 1]$ but is not bounded there.

1.4. Give an example of a function which is continuous on the interval $(-\infty, 0]$ but does not attain global maximum and global minimum.

1.5. On the interval $(0, 2)$ there exists a point c such that the tangent line to the graph of the function $y = x^3$ at the point (c, c^3) is parallel to the straight line passing through the points $(0, 0)$ and $(2, 8)$.
   (i). Explain without calculations why such point c necessarily exists.
   (ii). Find c.

1.6. Prove that if a nonconstant function $f(x)$ is continuous on the interval $[a, b]$ and differentiable on (a, b), then there exist points x_1 and x_2 on the interval (a, b) such that $f'(x_1) < 0$ and $f'(x_2) > 0$.

In the next problems prove the given inequality using the Mean Value Theorem.

1.7. $2\sqrt{x} > 3 - \frac{1}{x}$, for $x > 1$.

1.8. $\sin x < x$, for $x > 0$.

1.9. $\cos x > 1 - \frac{x^2}{2}$, for $x > 0$.

1.10. $\sin x > x - \frac{x^3}{6}$, for $x > 0$.

1.11. $\tan x > x$, for $0 < x < \frac{\pi}{2}$.

1.12. $e^x > 1 + x$, for $x > 0$.

1.13. $e^x > 1 + x + \frac{x^2}{2}$, for $x > 0$.

1.14. $e^x > 1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!}$, for $x > 0$. (Hint: use mathematical induction)

2. Factorization of Polynomials

2.1. **Complex Polynomials.** The set $\mathbb{R}$ of real numbers can be extended to a bigger set of the so-called *complex numbers*. This is done by introducing a single *imaginary* number $i = \sqrt{-1}$. Complex numbers can be written in the form $z = a + ib$, where $a, b \in \mathbb{R}$. In this representation a is called the *real* part of z, and b the *imaginary* part of z, denoted respectively by $\operatorname{Re} z$ and $\operatorname{Im} z$. Real numbers can be viewed as a subset of complex numbers with zero imaginary part. Thus, denoting the space of complex numbers by $\mathbb{C}$, we have the following chain of inclusions

$$\mathbb{N} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}.$$

We may extend the definition of arithmetic operations on real numbers to the space of complex numbers as follows:

(i) $(a + ib) + (a' + ib') = (a + a') + i(b + b')$
(ii) $(a + ib) \cdot (a' + ib') = (aa' - bb') + i(ab' + a'b)$
(iii) $\dfrac{a + ib}{a' + ib'} = \dfrac{aa' + bb'}{a'^2 + b'^2} + \dfrac{ba' - ab'}{a'^2 + b'^2} i$, if $a' + b'i \neq 0 = 0 + i0$.

One can verify that when $b = b' = 0$, the above formulas provide the usual operations of addition, multiplication and division for reals. Note that $i \cdot i = i^2 = -1$, which in particular means that the equation $z^2 + 1 = 0$ *over* the set of complex numbers has two complex roots: i and $-i$. This is in contrast with reals over which this equation has no solution.

With these operations on complex numbers we may define complex polynomials as functions on complex numbers defined by

$$P(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n, \quad \text{where} \quad a_j \in \mathbb{C}. \tag{5}$$

If $a_0 \neq 0$, the n is called the degree of $P(z)$. In particular, if we ignore the choice of the letter for the unknown variable (x vs. z), the usual polynomials with real coefficients are examples of complex polynomials. (In other words, a polynomial is called *real* if in (5), $a_j \in \mathbb{R}$ for all j.) The following theorem is usually known as the *Fundamental Theorem of Algebra*.

Theorem 2.1. *Suppose $P(z)$ is a complex polynomial of degree $n > 0$. Then $P(z)$ has exactly n complex roots.*

In this theorem the number of roots should be counted with *multiplicity*, in other words, some roots may have to be counted more than once. For example, $z^2 + 2z + 1 = (z + 1)^2 = 0$ has two roots both of which are $z = -1$. In general, if $w_1, w_2, \ldots, w_m$ are the distinct roots of a polynomial $P(z)$, then we can write

$$P(z) = a_0 (z - w_1)^{k_1} (z - w_2)^{k_2} \ldots (z - w_m)^{k_m}. \tag{6}$$

This is called a *factorization* of a complex polynomial into complex linear factors. It is unique up to a change of order. The exponent k_j is called the *multiplicity* of the root w_j. The Fundamental Theorem of Algebra implies that $k_1 + k_2 + \cdots + k_m = n$. The proof of Theorem 2.1 requires some knowledge of complex analysis, a branch of mathematics that studies functions of complex variables.

Example 2.1. Consider the equation $f(z) = z^4 + z^2 = 0$. According to the Fundamental Theorem of Algebra, $f(z)$ has four roots. These can be easily found. Indeed, $z^4 + z^2 = z^2(z^2 + 1)$. Thus the roots are $z = 0$ (counted twice), $z = i$ and $z = -i$. So

$$f(z) = z^2(z - i)(z + i)$$

is a factorization of this polynomial into linear factors. ⋄

Note that not every real polynomial admits a factorization into *real* linear factors (e.g., $x^2 + 1$).

2.2. **Factorization of Real Polynomials.** An important operation on complex numbers is *complex conjugation*, or just conjugation, which is denoted by a horizontal bar, and defined as follows:

$$\overline{a + ib} = a - ib.$$

In other words, to conjugate a complex number we simply change the sign of the imaginary part of the number. Note that if z is a real number, then $\overline{z} = z$, i.e., conjugation leaves real numbers unchanged.

Let $w = a + ib$ be a complex number. Then $\overline{w} = a - ib$. Consider the expression$(z - w)(z - \overline{w})$. Then

$$(z - w)(z - \overline{w}) = z^2 - wz - \overline{w}z + w\overline{w} = z^2 - (w + \overline{w})z + w\overline{w}. \tag{7}$$

We have $w + \overline{w} = (a + ib) + (a - ib) = 2a$, and $w\overline{w} = (a + ib)(a - ib) = a^2 + b^2$. Both are real numbers. Thus the product of two monomials as above with conjugate free terms yields a degree two polynomial with real coefficients.

Suppose now

$$P(z) = z^n + b_1 z^{n-1} + \cdots + b_{n-1} z + b_n, \quad \text{where} \quad b_j \in \mathbb{R}, \tag{8}$$

is a polynomial of degree n with real coefficients, and let ζ be a complex root of $P(z)$. Then

$$\zeta^n + b_1 \zeta^{n-1} + \cdots + b_{n-1}\zeta + b_n = 0.$$

Conjugation of both sides of this equation gives

$$\overline{\zeta}^n + b_1 \overline{\zeta}^{n-1} + \cdots + b_{n-1}\overline{\zeta} + b_n = 0.$$

Note that coefficients b_j did not change because conjugation does not change real numbers. We also used here the fact that for $z, w \in \mathbb{C}$, we have $\overline{z + w} = \overline{z} + \overline{w}$, and $\overline{z \cdot w} = \overline{z} \cdot \overline{w}$, which can be verified directly. What the last equation tells us is that $\overline{\zeta}$ is also a root of $P(z)$. In other words, if ζ is a complex root of $P(z)$ and ζ is not a real number, then $\overline{\zeta}$ is also a root of $P(z)$. Thus we may write

$$P(z) = (z - x_1)(z - x_2) \cdots (z - x_m)(z - z_1)(z - \overline{z_1}) \cdots (z - z_k)(z - \overline{z_k}), \tag{9}$$

where $x_1, \ldots, x_m$ are the real roots of $P(x)$, and $z_1, \overline{z_1}, \ldots, z_k, \overline{z_k}$ are the pairs of complex roots and their conjugates. Using the calculation in (7) we have

$$(z - z_1)(z - \overline{z_1}) = z - A_1 z + B_1, \quad \text{where} \quad A_1 = 2\mathrm{Re}\, z_1, \text{ and } B_1 = (\mathrm{Re}\, z_1)^2 + (\mathrm{Im}\, z_1)^2,$$

and similarly for the other pairs of complex conjugate roots of $P(z)$. Using this, and replacing z with x in (9) yields

$$P(x) = (x - x_1)(x - x_2) \cdots (x - x_m)(x^2 - A_1 x + B_1)(x^2 - A_2 x + B_2) \cdots (x^2 - A_k x + B_k),$$

where all the coefficients are real numbers. Thus, we proved the following theorem.

Theorem 2.2. *Suppose $P(x)$ is a real polynomial of degree $n > 0$. Then $P(x)$ admits factorization into a product of linear and quadratic factors with real coefficients.*

This theorem is used in the theory of integration of rational functions using partial fractions.

Example 2.2. Let $P(x) = x^4 + 1$. This polynomial does not have any real roots. Nevertheless, according to Theorem 2.2, it can be factored into a product of two real polynomials. But what are these? One possible solution would be to find complex roots of $P(z)$ and then to multiply the conjugate monomials as discussed above. However, finding complex roots is not an easy task. Instead, we can try to factorize $P(z)$ into two polynomials $x^2 + ax + 1$ and $x^2 + bx + 1$ for some $a, b \in \mathbb{R}$. We get

$$(x^2 + ax + 1)(x^2 + bx + 1) = x^4 + ax^3 + x^2 + bx^3 + abx^2 + bx + x^2 + ax + 1.$$

We set this equal to $x^4 + 1$ and compare the coefficients of x^3, x^2, and x. It follows that $a = -b$ and $ab = -2$. So we may take $a = \sqrt{2}$ and $b = -\sqrt{2}$. This gives the required factorization:

$$x^4 + 1 = (x^2 + \sqrt{2}\,x + 1)(x^2 - \sqrt{2}\,x + 1).$$

◇

3. The Gamma function

The Gamma function $\Gamma(x)$ is defined as an improper integral

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt. \tag{10}$$

This function brings together integration by parts and improper integrals. It can be seen as a solution to the following interpolation problem: find a smooth curve that connects the points (x, y) in the plane given by $y = 1 \cdot 2 \cdot 3 \cdot \dots \cdot x = x!$ at the positive integer values for x.

A plot of the first few factorials (see Fig 1.) makes clear that such a curve can be drawn, but it would be preferable to have a formula that precisely describes the curve, in which the number of operations does not depend on the size of n. The formula for the factorial $n!$ cannot be used directly for fractional values of n since it is only valid when n is a positive integer. There is, in fact, no such simple solution for factorials. Any combination of sums, products, powers, exponential functions or logarithms with a fixed number of terms will not suffice to express $n!$. But it is possible to find a general formula as an integral depending on a parameter. This was discovered by L. Euler in 1729.[3]

First consider the case $x = 1$. We have

$$\Gamma(1) = \int_0^\infty e^{-t} dt = \lim_{s\to\infty} \int_0^s e^{-t} dt = \lim_{s\to\infty} -e^{-t}\Big|_0^s = 1.$$

Further, using integration by parts, one can show that $\Gamma(n + 1) = n \cdot \Gamma(n)$. Indeed, for an integer $n \geq 1$,

$$\Gamma(n + 1) = \int_0^\infty t^{n+1-1} e^{-t} dt = \int_0^\infty t^n e^{-t} dt.$$

Consider the indefinite integral $\int t^n e^{-t} dt$. We apply integration by parts by choosing $u = t^n$, and $dv = e^{-t}dt$. Then $du = n\,t^{n-1}dt$ and $v = -e^{-t}$. According to the integration by parts formula, we have

$$\int t^n e^{-t} dt = -t^n\, e^{-t} - \int -e^{-t} n\, t^{n-1} dt = -t^n\, e^{-t} + n \int t^{n-1} e^{-t} dt.$$

[3]The symbol $\Gamma(x)$ and the name were proposed in 1814 by A.M. Legendre.

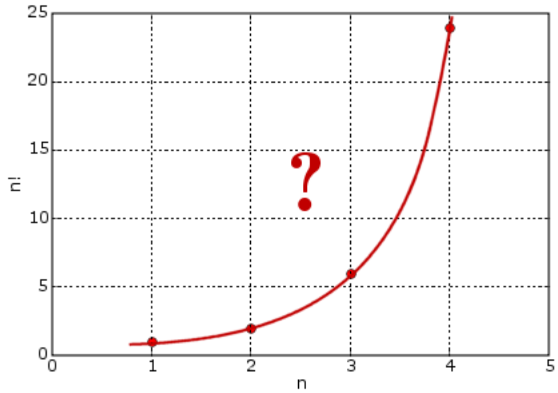

FIGURE 4. Interpolating $n!$

Thus,

$$\int_0^{\infty} t^n e^{-t} dt = \lim_{s\to\infty} \left[-t^n\, e^{-t} \Big|_0^s + n \int_0^s t^{n-1} e^{-t} dt \right] \tag{11}$$

For the first term inside the limit above we get

$$\lim_{s\to\infty} \left(\frac{-t^n}{e^t}\right)\Bigg|_0^s = \lim_{s\to\infty} -\frac{s^n}{e^s}.$$

Using L'Hôpital's Rule n times we see that

$$\lim_{s\to\infty} \frac{s^n}{e^s} = \lim_{s\to\infty} \frac{n!\, s^0}{e^s} = 0.$$

For the second term in the right hand side of (11) we have

$$\lim_{s\to\infty} n \int_0^s t^{n-1} e^{-t} dt = n\Gamma(n).$$

Combining everything together we have $\Gamma(n+1) = n\Gamma(n)$. This identity provides a reduction formula which can be used to compute inductively the values of the Gamma function for positive integers:

$$\Gamma(n+1) = n! \quad \text{where} \quad n \in \mathbb{N}.$$

Indeed, $\Gamma(2) = 1$; $\Gamma(3) = \Gamma(2+1) = 2\cdot\Gamma(2) = 2$; $\Gamma(4) = \Gamma(3+1) = 3\cdot\Gamma(3) = 3\cdot 2$, etc.

In fact, by inspection we see that our application of the integration by parts formula is valid not only for integer values n, but for all real $x > 0$ (see Exercises 3.1 and 3.2 for the case $0 < x < 1$), and so we have

$$\Gamma(x+1) = x\,\Gamma(x) \quad \text{for all} \quad x > 0. \tag{12}$$

The graph of the Gamma function is given on Figure 5.

Exercises

3.1. For $x \geq 1$ the above calculations show the convergences of the improper integral that defines the Gamma function. However, if $x < 1$, then the integral in (10) contains a negative power of t ($x-1$ becomes negative). Use the comparison test for improper integrals to show that the Gamma function is well-defined for $0 < x < 1$. (*Hint:* split the integral in (10) into two integrals.)

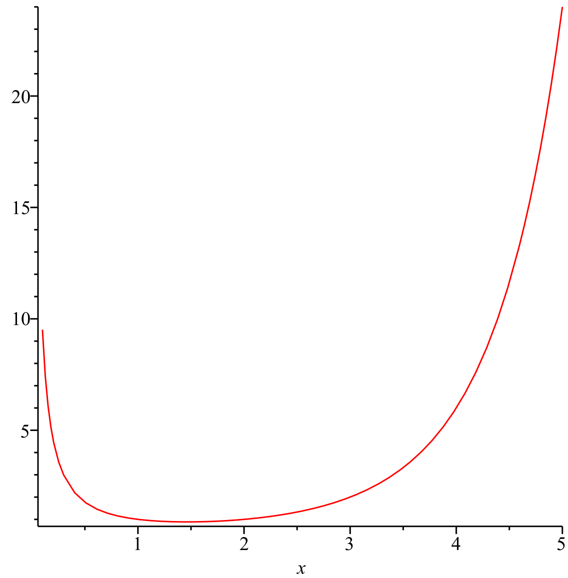

FIGURE 5. The graph of $\Gamma(x)$

3.2. Verify formula (12) for the case when $0 < x < 1$.
3.3. Show that the integral in (10) diverges if $x \leq 0$.
3.4. The integral

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

is called the *Gaussian* integral. It is particularly important in probability theory and statistics. Use the value of this integral to evaluate $\Gamma(1/2)$.
3.5. Use Problem 3.4 to calculate $\Gamma(5/2)$.
3.6. Prove that $\lim_{x \to 0^+} \Gamma(x) = +\infty$.

4. SEQUENCES

4.1. Convergence of sequences.

Definition 4.1. *A sequence s is a function $s : \mathbb{N} \to \mathbb{R}$. It can be thought of as a list of numbers*

$$s_1, s_2, s_3, \ldots,$$

where $s_n = s(n)$ for $n \in \mathbb{N}$.

Example 4.1.

(i) $\{s_n\} = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\right\}$. The corresponding function $s : \mathbb{N} \to \mathbb{R}$ is given by $s(n) = \frac{1}{n}$.

(ii) Let

$$\{s_n\} = \left\{\frac{1}{2}, \frac{1}{6}, \frac{1}{12}, \frac{1}{20}, \ldots\right\}.$$

Here $s(n) = \frac{1}{n \cdot (n+1)}$.

(iii) $\{s_n\} = \{1, -1, 1, -1, 1, -1 \ldots\}$. For this sequence we can take, for example, $s_n = (-1)^{n+1}$.

◇

A sequence is defined *inductively* (or *recursively*) if $s_n = f(s_1, \dots s_{n-1})$, i.e., each term of the sequence is defined as a function of previously defined terms.

Example 4.2. (Fibonacci sequence[4].) By definition, the first two terms of the Fibonacci sequence $\{f_n\}$ are 1 and 1, and each consequent number is the sum of the previous two. Inductively this can be defined as follows.

$$f_1 = f_2 = 1, \quad f_n = f_{n-1} + f_{n-2}, \quad \text{for } n > 2.$$

The first several terms of the Fibonacci sequence can be easily computed to be

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \dots.$$

◇

Example 4.3. $s_1 = \sqrt{2}$, $s_n = \sqrt{2 + s_{n-1}}$ for $n > 1$. Then

$$s_1 = \sqrt{2}, s_2 = \sqrt{2+\sqrt{2}}, s_3 = \sqrt{2+\sqrt{2+\sqrt{2}}}, s_4 = \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}, \dots$$

◇

Definition 4.2. *A sequence $\{s_n\}$ converges to a real number L if for every positive number ϵ, there exists an $N \in \mathbb{N}$ such that whenever $n > N$ it follows that $|s_n - L| < \epsilon$. In this case we write*

$$\lim_{n\to\infty} s_n = L.$$

If $\{s_n\}$ does not converge, it is said to diverge.

The above definition is sometimes called the ϵ-N definition of convergence of a sequence.

Example 4.4. $\lim\limits_{n\to\infty} \dfrac{1}{\sqrt{n}} = 0.$

To prove this, set $s_n = \frac{1}{\sqrt{n}}$, and $L = 0$. We need to show that given any $\epsilon > 0$, there exists an index $N > 0$ such that $|s_n - L| = |1/\sqrt{n}| < \epsilon$ for $n > N$. The inequality $1/\sqrt{n} < \epsilon$ is equivalent to $n > 1/\epsilon^2$. By taking $N = \lceil 1/\epsilon^2 \rceil$, we ensure that if $n > N$, then $|1/\sqrt{n}| < \epsilon$. (Recall that $\lceil x \rceil$ is the *ceiling* function; it equals the smallest integer bigger than or equal to x.) ◇

Using a similar argument one can show that $\lim\limits_{n\to\infty} \dfrac{1}{n^p} = 0$ for $p > 0$ (Exercise 4.1(i)).

Example 4.5. $\lim\limits_{n\to\infty} \dfrac{n+1}{n} = 1.$

Let $s_n = \frac{n+1}{n}$, and $L = 1$. Then

$$|s_n - L| = \left| \frac{n+1}{n} - 1 \right| = \left| \frac{1}{n} \right| < \epsilon,$$

and therefore, the choice of $N = \lceil 1/\epsilon \rceil$ will ensure that $|s_n - L| < \epsilon$. ◇

Example 4.6. $\lim\limits_{n\to\infty} \left(\dfrac{1}{2}\right)^n = 0.$

Set $s_n = \left(\frac{1}{2}\right)^n$, $L = 0$. Then

$$\left| \left(\frac{1}{2}\right)^n \right| < \epsilon \iff n \ln(1/2) < \ln \epsilon \iff n > \frac{\ln \epsilon}{\ln(1/2)}.$$

[4]The Fibonacci sequence is named after Leonardo of Pisa, who was known as Fibonacci (a contraction of filius Bonaccio, "son of Bonaccio"). Fibonacci's 1202 book *Liber Abaci* introduced the sequence to Western European mathematics, although the sequence had been previously described in Indian mathematics.

Note that $\ln(1/2) < 0$, and so division by this number reverses the inequality. We could take $N = \lceil \frac{\ln \epsilon}{\ln(1/2)} \rceil$, but then N becomes negative for $\epsilon > 1$. So a better choice is

$$N = \max\left\{ \left\lceil \frac{\ln \epsilon}{\ln(1/2)} \right\rceil, 1 \right\}$$

$\diamond$

Definition 4.3. $\lim_{n\to\infty} s_n = \infty$ *means that for any real number* M *there exists an* $N \in \mathbb{N}$ *such that* $s_n > M$ *whenever* $n \geq N$.

Example 4.7. The Fibonacci sequence diverges to infinity. Indeed, starting with $n = 5$ we see that $f_n \geq n$. Therefore, given any number $M > 0$, $f_n > M$ for all $n > \lceil M \rceil$. $\diamond$

Example 4.8. Investigate convergence of $\{r^n\}$ for different values of $r > 0$.

Suppose $r > 1$. Then if $M > 0$ is arbitrary, the inequality $r^n > M$ is satisfied for $n > \dfrac{\ln M}{\ln r}$. Thus r^n diverges to infinity if $r > 1$. If $r = 1$, then r^n is a constant sequence 1, hence converges to 1. Finally, if $0 < r < 1$, then $\lim_{n\to\infty} r^n = 0$. Indeed, given $\epsilon > 0$, for $n > \max\left\{\dfrac{\ln \epsilon}{\ln r}, 1\right\}$ the inequality $r^n < \epsilon$ holds. $\diamond$

4.2. **Properties of sequences.** The following theorem provides a convenient way of calculating the limit by reducing the problem to algebraic manipulation of existing limits. It can be proved directly using the ϵ-N definition of convergence.

Theorem 4.4 (Algebraic Limit Theorem)**.** *If* $\lim a_n = A$, $\lim b_n = B$, *then*

(i) $\lim(ca_n) = cA$ *for* $c \in \mathbb{R}$,
(ii) $\lim(a_n + b_n) = A + B$,
(iii) $\lim(a_n \cdot b_n) = A \cdot B$,
(iv) $\lim\left(\dfrac{a_n}{b_n}\right) = \dfrac{A}{B}$, *if* $b_n \neq 0$ *and* $B \neq 0$.

Example 4.9. Examples of use of the Algebraic Limit Theorem.

1. $\lim_{n\to\infty} \dfrac{5n - 3n^2}{2n^2 + (-1)^n} = \lim_{n\to\infty} \dfrac{5/n - 3}{2 + \frac{(-1)^n}{n^2}} = -\dfrac{3}{2}$. Here we use the result of Exercise 4.1(i) and also the fact that $\lim_{n\to\infty} |a_n| = 0$ implies $\lim_{n\to\infty} a_n = 0$, which follows directly from the ϵ-N definition of convergence.
2. $\lim_{n\to\infty} \dfrac{n \ln n}{(n+1)^2} = \lim_{n\to\infty} \dfrac{n}{n+1} \cdot \lim_{n\to\infty} \dfrac{\ln n}{n+1} = 1 \cdot \lim_{n\to\infty} \dfrac{1/n}{1} = 0$. Here we used l'Hôpital's rule.

$\diamond$

A useful reduction for computing limits of sequences is the following: if f is a continuous function and $\{s_n\}$ is a sequence that converges to limit L, then $\lim_{n\to\infty} f(s_n) = f(L)$. Using this fact, one can prove that if a sequence is defined by an inductive formula

$$s_{n+1} = f(s_n), \tag{13}$$

where f is a continuous function, then assuming that the limit L of the sequence $\{s_n\}$ exists, it can be often found by taking the limit in (13): $L = f(L)$. Observe that $\lim s_n = \lim s_{n+1} = L$.

Example 4.10. Let $\{s_n\}$ be defined inductively by $s_1 = 1$, and

$$s_{n+1} = \frac{2s_n + 3}{4} \tag{14}$$

Assume that the limit of $\{s_n\}$ exists, say, $\lim_{n\to\infty} s_n = L$. Then we can take the limit as $n \to \infty$ on both sides of (14). We get

$$\lim_{n\to\infty} s_{n+1} = \lim_{n\to\infty} \frac{2s_n + 3}{4}.$$

it follows that

$$L = \frac{2L+3}{4}, \quad \text{so} \quad L = 3/2.$$

Hence, $\lim s_n = 3/2$. $\diamond$

If *a priori* it is not known that the limit of $\{s_n\}$ exits, then the calculation of L from equation (13) may produce unpredictable results, see Exercise 4.4 for details. Thus, justification of existence of the limit becomes an important problem on its own.

Theorem 4.5 (Squeeze Theorem)**.** *If $a_n \le b_n \le c_n$ for $n > n_0$ and*

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L,$$

then $\lim_{n\to\infty} b_n = L$.

Proof. Take any $\epsilon > 0$. We need to find $N > 0$ such that $|b_n - L| < \epsilon$ whenever $n > N$. Since $a_n \to L$, there is $N_1 > 0$ such that for $n > N_1$ we have $|a_n - L| < \epsilon$. An equivalent form of this inequality is

$$L - \epsilon < a_n < L + \epsilon.$$

Similarly, since $c_n \to L$, there is $N_2 > 0$ such that $|c_n - L| < \epsilon$ for $n > N_2$, or

$$L - \epsilon < c_n < L + \epsilon.$$

Take $N = \max\{N_1, N_2\}$. Then for $n > N$ we have

$$L - \epsilon < a_n \le b_n \le c_n < L + \epsilon,$$

which implies that $|b_n - L| < \epsilon$. □

Example 4.11. $\lim_{n\to\infty} \frac{5^n}{n^n} = 0$.
To prove this we use the Squeeze Theorem. Indeed, for $n > 6$,

$$0 < \frac{5^n}{n^n} = \left(\frac{5}{n}\right)^n < \left(\frac{5}{6}\right)^n.$$

We may take $a_n = 0$, $b_n = \frac{5^n}{n^n}$, and $c_n = \left(\frac{5}{6}\right)^n$. Since $\lim_{n\to\infty} \left(\frac{5}{6}\right)^n = 0$ by Example 4.8, the Squeeze Theorem implies that $\lim_{n\to\infty} \frac{5^n}{n^n} = 0$. $\diamond$

Definition 4.6. *A sequence $\{s_n\}$ is called increasing if $s_{n+1} \ge s_n$ for all n, strictly increasing if $s_{n+1} > s_n$ for all n. Decreasing and strictly decreasing sequences are defined similarly. Decreasing and increasing sequences are called* monotone *sequences.*

Example 4.12. $\left\{\frac{n}{n^2+1}\right\}$ is a decreasing sequence. This can be proved either by verifying the inequality $\frac{n+1}{(n+1)^2+1} \ge \frac{n}{n^2+1}$ for all n, or by showing that that function $f(x) = \frac{x}{x^2+1}$ has a negative derivative for $x > 1$. $\diamond$

4.3. **Least upper bound, monotone convergence.**

Definition 4.7. *An upper bound of a non-empty subset S of $\mathbb{R}$ is a number b such that $b \geq s$, for any $s \in S$. A number l is a least upper bound or supremum of S, denoted by* $\sup S$*, if l is an upper bound of S, and if b is another upper bound of S then $l \leq b$.*

Example 4.13.

(1) $S_1 = \{0, 1/2, 2/3, 3/4, \dots\}$. Then $\sup S_1 = 1$.
(2) $S_2 = \mathbb{N}$. This set is unbounded, and therefore, the upper bound for this set does not exist.
(3) Let
$$S_3 = \{\sin n,\ n \in \mathbb{N}\} = \{\sin 1, \sin 2, \sin 3, \dots\}.$$
This set is bounded above by 1, since $\sin x \leq 1$ for any x. But is there $\sup S_3$? If n could attain any real value, then since $\sin(\frac{\pi}{2} + 2\pi k) = 1$, the supremum would be 1. However, since $n \in \mathbb{N}$, $\sin n \neq 1$ for any n. Therefore, if $\sup S_3$ exists, in order to find it, one needs to investigate how close a natural number n can come to the set of numbers of the form $\frac{\pi}{2} + 2\pi k, k \in \mathbb{N}$.

◇

Lower bound and greatest lower bound (*infimum*) are defined similarly.

Definition 4.8. *A sequence $\{s_n\}$ is bounded above (below) if the set*
$$\{s_n; n \in \mathbb{N}\} = \{s_1, s_2, s_3, \dots\}$$
has an upper (lower) bound.

Axiom of Completeness. *Every nonempty set of real numbers that has an upper bound, has a least upper bound.*

The Axiom of Completeness distinguishes real numbers from rational numbers. For example, the set $S = \{x \in \mathbb{R} : x^2 < 2\}$ has a least upper bound $\sqrt{2}$. However, the set of rational numbers r, such that $r^2 < 2$, is bounded, but it does not have a least upper bound in $\mathbb{Q}$ ($\sqrt{2}$ is not rational!). Thus, the Axiom of completeness is false for rationals.

Let us return to Example 4.13(3). Since the set S_3 is bounded above by 1, the Axiom of Completeness guarantees that S_3 has a supremum, although it is a non-trivial problem to determine what it is.

Theorem 4.9 (Monotone Convergence Theorem)**.** *Every bounded monotone sequence converges.*

Proof. Consider the case when $\{s_n\}$ is an increasing sequence bounded above. Since the set $S = \{s_n;\ n \in \mathbb{N}\}$ is bounded, by the Axiom of Completeness, there exists $l = \sup S$. We claim that l is the limit of $\{s_n\}$. Indeed, take any $\epsilon > 0$. Then since l is the supremum of S, there exits an index N such that $s_N > l - \epsilon$. But since the sequence is increasing, we have $s_n > l - \epsilon$ for all $n > N$. This means that $|l - s_n| < \epsilon$ for $n > N$, which proves that $\lim s_n = l$.

The case when $\{s_n\}$ is decreasing and bounded below can be proved in a similar way. □

Example 4.14. Consider the sequence defined in Example 4.3. We may use induction to show that $s_n < 2$ for all n. Indeed, $s_1 = \sqrt{2} < 2$. If $s_n < 2$, then $2 + s_n < 4$. Taking the square root on both sides, we get $\sqrt{2 + s_n} < 2$, which means that $s_{n+1} < 2$. This shows that the inequality $s_n < 2$ holds for all n.

Further, $\{s_n\}$ is increasing. Indeed, $s_n < \sqrt{2 + s_n}$ is equivalent to $s_n^2 - s_n - 2 < 0$, which holds true for $-1 < s_n < 2$. By the previous paragraph $s_n < 2$, and therefore, $s_n < s_{n+1}$ for all n.

Thus, $\{s_n\}$ is a bounded monotone sequence, and by the Monotone Convergence Theorem $\{s_n\}$ converges. The limit L can be found by taking the limit as $n \to \infty$ on both sides of $s_n = \sqrt{s_n + 2}$. We have

$$L = \sqrt{2 + L} \implies L^2 - L - 2 = 0.$$

This equation has two roots: -1 and 2. Since $s_n > 0$ for all n, $L = 2$. ⋄

Exercises

4.1. Using only Definition 4.1 prove

(i) $\lim_{n\to\infty} \dfrac{1}{n^p} = 0, \ \ p > 0.$

(ii) $\lim_{n\to\infty} \dfrac{1+2n}{5+3n} = \dfrac{2}{3}.$

(iii) $\lim_{n\to\infty} \dfrac{\sin n}{n+1} = 0.$

4.2. Give the definition of divergence of a sequence without referring to converge of a sequence. Use your definition to show that the sequence $s_n = (-1)^n + \frac{1}{n}$ diverges.

4.3. Give a definition of $\lim_{n\to\infty} s_n = -\infty$. Use your definition to verify that $\lim \log_a n = -\infty$ for $0 < a < 1$.

4.4. Let the sequence $\{s_n\}$ be defined inductively as $s_1 = 1$, and $s_{n+1} = {s_n}^2 - 1$ for $n > 1$. Compute L using the ideas of Example 4.10, and then show that this L cannot be the limit of the sequence s_n.

4.5. Use the Squeeze Theorem to find $\lim\limits_{n\to\infty} \dfrac{\sin n + \cos n}{\sqrt{n}}$.

4.6. Determine without proof $\sup S$, the *supremum* of the set S given by

$$S = \left\{ \frac{n}{n+m}, \text{ where } n, m \in \mathbb{N} \right\}.$$

4.7. Prove that if a sequence $\{s_n\}$ converges, then the set $S = \{s_1, s_2, \dots\}$ is bounded.

4.8. Let $\{s_n\}$ be defined as $s_1 = 0.3$, $s_2 = 0.33$, $s_3 = 0.333$, ... Prove that $\{s_n\}$ converges.

4.9. Let $\{f_n\}$ be the Fibonacci sequence as defined in Example 4.2. Consider a sequence

$$s_1 = 1, \quad s_n = \frac{f_{n+1}}{f_n} \text{ for } n > 1.$$

Assume that s_n converges. Find its limit.

4.10. Show that the sequence $\{x_n\}$ defined by $x_1 = 3$, $x_{n+1} = \frac{1}{4 - x_n}$ for $n > 1$, converges. Then find the limit.

5. Taylor Series

Let $f(x)$ be a function that has derivatives of all orders on the interval $(a-R, a+R)$ for some $a \in \mathbb{R}$, and $R > 0$. Suppose that $f(x)$ can be represented on $(a-R, a+R)$ by a convergent power series

$$\sum_{n=0}^{\infty} c_n (x-a)^n. \tag{15}$$

This means that for any $x \in (a-R, a+R)$, the series (15) converges to $f(x)$. Then by direct differentiation of the power series (15), we see that $f^{(n)}(a) = n!\, c_n$, for all $n > 0$ (here $f^{(n)}$ denotes the derivative of $f(x)$ of order n). From this we conclude that

$$c_n = \frac{f^{(n)}(a)}{n!},$$

and thus the series in (15) becomes

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots \tag{16}$$

This is called the *Taylor series centred at* $x = a$ *associated with* $f(x)$. If $a = 0$, then (16) becomes

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots, \tag{17}$$

which is called the *Maclaurin series* associated with $f(x)$.

Example 5.1. Let $P(x)$ be a polynomial of degree N,

$$P(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_N x^N.$$

By inspection, $c_n = \dfrac{P^{(n)}(0)}{n!}$ for $n = 1, \dots N$, and $c_n = 0$ for $n > N$. Thus, the Maclaurin series associated with $P(x)$ is exactly $P(x)$.

In general, however, one cannot immediately conclude that the Taylor or Maclaurin series associated with a function $f(x)$ converges to $f(x)$. In fact, it is not even clear whether the Taylor series of a given function converges at all. (Note that when we derived (16) we assumed to begin with that $f(x)$ has a power series representation.) Define the *Taylor polynomial* to be

$$T_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(N)}(a)}{N!}(x-a)^N, \tag{18}$$

i.e., $T(x)$ is simply the order N partial sum of the Taylor series (16). Thus, by the definition of convergence, in order to show the convergence of the Taylor series to $f(x)$ we need to show that

$$\lim_{N\to\infty} T_N(x) = f(x) \tag{19}$$

for all x on some interval. If we define the *remainder* of the Taylor series to be

$$R_N(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n - T_N(x) = \frac{f^{(N+1)}(a)}{(N+1)!}(x-a)^{N+1} + \frac{f^{(N+2)}(a)}{(N+2)!}(x-a)^{N+2} + \dots, \tag{20}$$

then proving (19) is equivalent to showing

$$R_N(x) \to 0, \quad \text{as} \quad N \to \infty.$$

The following theorem provides a useful tool for proving convergence of Taylor series. For simplicity, we consider the case when $a = 0$. Then $T_N(x) = f(0) + f'(0)x + \dots \frac{f^{(N)}(0)}{N!}x^N$, and $R(x) = \frac{f^{(N+1)}(0)}{(N+1)!}x^{N+1} + \dots$.

Theorem 5.1 (Lagrange's Remainder Theorem)**.** *Let f be infinitely differentiable on $(-R, R)$. Then there exists a number c satisfying $|c| < |x|$ such that*

$$R_N(x) = \frac{f^{(N+1)}(c)}{(N+1)!}x^{N+1}. \tag{21}$$

Example 5.2. Let $f(x) = e^x$. Then $f^{(n)}(0) = e^0 = 1$ for all n. Therefore, $c_n = \frac{1}{n!}$, and we have

$$e^x \sim \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots.$$

The remainder of order N of this Maclaurin series is

$$R_N = \frac{x^{N+1}}{(N+1)!} + \frac{x^{N+2}}{(N+2)!} + \dots.$$

According to Lagrange's Remainder Theorem, there is a number c, $|c| < |x|$, such that

$$R_N(x) = \frac{f^{(N+1)}(c)}{(N+1)!}x^{N+1} = \frac{e^c}{(N+1)!}x^{N+1}.$$

For any fixed x, $R_N(x) \to 0$, since for any x, $\frac{x^n}{n!} \to 0$ as $n \to \infty$. Thus

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots.$$

for all $x \in \mathbb{R}$.

Example 5.3. Let

$$g(x) = \begin{cases} e^{-1/x^2}, & \text{if } x > 0 \\ 0, & \text{if } x \leq 0 \end{cases} \tag{22}$$

Since e^{-1/x^2} approaches 0 as $x \to 0$, the function $g(x)$ is continuous at 0. In fact, using L'Hôpital's Rule one can show that $g(x)$ has continuous derivatives of any order at $x = 0$, and $g^{(n)}(0) = 0$ for any $n > 0$. The Maclaurin series associated to $g(x)$ is, therefore, identically zero. It follows that the Maclaurin series associated with $g(x)$ does not converge to $g(x)$ for $x > 0$.

Definition 5.2. *An infinitely differentiable function $f(x)$ is called real-analytic in a neighbourhood of a point $x = a$, if for some positive R the Taylor series* (16) *associated with $f(x)$ converges to $f(x)$ on $(a - R, a + R)$.*

Thus, e^x is a real-analytic function, while the function $g(x)$ in Example 5.3 is not real analytic near $x = 0$.

Proof of Lagrange's Remainder Theorem. . First note the following version of the Mean Value Theorem: If $f(x)$ and $g(x)$ are continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b) and $g'(x) \neq 0$, then there exists a point $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)} \tag{23}$$

This can be proved by applying the Mean Value Theorem to the function $h(x) = (f(b)-f(a))g(x)-(g(b)-g(a))f(x)$.

Note that the n-th order derivative of $R_N(x)$ at $x=0$ vanishes for $n=0,1,2,\dots,N$. Therefore, if we apply (23) to functions $f(x) = R_N(x)$ and $g(x) = x^{N+1}$, then (assume $x>0$ for simplicity) there exists a point $x_1 \in (0,x)$ such that

$$\frac{R_N(x)}{x^{N+1}} = \frac{R'_N(x_1)}{(N+1)x_1^N}.$$

We now repeat the process and apply (23) to functions $f(x) = R'_N(x)$ and $g(x) = x^N$ on the interval $(0,x_1)$: there is $x_2 \in (0,x_1)$ such that

$$\frac{R'_N(x_1)}{x_1^N} = \frac{R''_N(x_2)}{N x_2^{N-1}}.$$

Continue the process inductively N times. In the end we get

$$R_N(x) = \frac{x^{N+1}}{(N+1)!}\frac{R_N^{(N+1)}(x_{N+1})}{x_{N+1}^{N-N}},$$

where $x_{N+1} \in (0,x_N) \subset \dots \subset (0,x)$. Now set $c = x_{N+1}$, then $c^{N-N}=1$, and we can write

$$R_N(x) = \frac{R_N^{(N+1)}(c)}{(N+1)!}x^{N+1} = \frac{f^{(N+1)}(c)}{(N+1)!}\,x^{N+1},$$

where the last equality follows from the fact that $R_N^{(N+1)}(x) = (f(x)-T_N(x))^{(N+1)} = f^{(N+1)}(x)$, because $T_N^{(N+1)} \equiv 0$. This proves the theorem. □

Example 5.4. Let $f(x) = (1+x)^{1/2}$. Then

$$f^{(n)}(0) = \frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\dots\left(\frac{1}{2}-n+1\right).$$

Therefore,

$$c_n = \binom{1/2}{n} = \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\dots\left(\frac{1}{2}-n+1\right)}{n!},$$

and hence

$$(1+x)^{1/2} \sim \sum_{n=0}^{\infty}\binom{1/2}{n}x^n$$

is the associated Maclaurin series. This is called the *binomial series*. Let us try use Lagrange's Remainder Theorem again to determine convergence of the series above. We have

$$R_N(x) = \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\dots\left(\frac{1}{2}-N\right)(1+c)^{1/2-N}}{(N+1)!}x^{N+1}$$

for some c, $|c|<|x|$. If $|x|<1$, then clearly $x^{N+1}\to 0$ as $N\to\infty$. Also, $\lim_{N\to\infty}\binom{1/2}{N} = 0$ (see Exercise 5.3). If $c>0$, then we also have $(1+c)^{1/2-N}\to 0$ as $N\to\infty$. However, if $c<0$, then $(1+c)^{1/2-N}$ does not go to zero, and we cannot be sure that $R_N(x)$ goes to zero.

In general, the binomial series converges for $x\in(-1,1)$, and we have

$$(1+x)^k = \sum_{n=0}^{\infty}\binom{k}{n}x^n, \quad k\in\mathbb{R}, \quad \text{and } |x|<1.$$

Exercises

5.1. Show that the function $g(x)$ in Example 5.3 satisfies $g'(0) = 0$.

5.2. Find the Maclaurin series for $f(x)$. Find the radius of convergence of the series, and show, using Lagrange's remainder theorem that the series converges to $f(x)$.

(i) $f(x) = \cos x$
(ii) $f(x) = \sin 2x$.
(iii) $f(x) = e^{2x}$.

5.3. Show that for any m,

$$\lim_{n\to\infty} \binom{m}{n} = 0.$$

5.4. Compute $\displaystyle\sum_{n=0}^{\infty} n(0.5)^n$.

5.5. Suppose that the function $f(x)$ can be represented by a power series

$$f(x) = \sum_{n=0}^{\infty} \frac{(x+1)^n}{2^n}.$$

Find the first two terms of the Taylor series of $f(x)$ centred at $x = 0$. (Hint: use the previous problem).

5.6. Evaluate the integral

$$\int_0^1 \frac{\ln(1-x)}{x} dx.$$

Hint: Use Taylor series expansion and the identity $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

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