COMPLEX ANALYSIS I, MATH 4156/9056, WINTER 2010

HOMEWORK ASSIGNMENT 3.

Due February 25.

3.1. Find an explicit formula for the Carathéodory distance $C_D(a, b)$ in the case when D is a disc

$$D = \{z \in \mathbb{C} : |z - z_0| < r\},$$

where $r > 0$ and $z_0 \in \mathbb{C}$.

3.2. Prove that if two functions u and v harmonic on a domain D coincide on an open set $U \subset D$, then $u \equiv v$.

3.3. Suppose that $f(z) = u(z) + iv(z)$ is analytic for $|z| < 1$ and that $u(z)$ extends to be continuous on the closed disc $\{|z| \leq 1\}$. Show that

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} u(e^{i\phi}) \frac{e^{i\phi} + z}{e^{i\phi} - z} d\phi + iv(0), \;\; |z| < 1.$$

3.4. Suppose $\{u_n\}$ is a uniformly bounded sequence of harmonic functions on $\mathbb{D}$. Prove that there exists a subsequence $\{u_{n_\nu}\}$ that converges uniformly on compact subsets of $\mathbb{D}$.

3.5. (For 9056 only.) Give an example of a non-constant function u harmonic in

$$U = \{x^2 + y^2 < x\},$$

continuous on $\overline{U} \setminus \{0\}$, and vanishing on $\partial U \setminus \{0\}$. (This shows, in particular, that the solution to Dirichlet problem becomes non-unique if you lift the boundary value condition just at one point.)