COMPLEX ANALYSIS I, MATH 4156/9056, WINTER 2010

HOMEWORK ASSIGNMENT 5.

Due March 25

5.1. Let $\{f_n\}$ be a sequence of rational functions that converges normally to $f(z)$ on the extended complex plane $\mathbb{C}^*$. Show that $f_n(z)$ has the same degree as $f(z)$ for large n. [Recall that $\deg f$, the *degree* of a rational function $f(z) = \frac{p(z)}{q(z)}$, is defined as $\max\{\deg p, \deg q\}$, where $p(z)$ and $q(z)$ are polynomials. It is known that if $\deg f = d$, then each value $w \in \mathbb{C}$, $w \neq f(\infty)$, is assumed exactly d times (counting multiplicities).]

5.2. Show that the function

$$e^{\frac{1}{z}} + e^{-\frac{1}{z}}$$

omits only the value ∞ at $z = 0$.

5.3. Let $f(z)$ be analytic on the punctured disc $\{0 < |z| < 1\}$ and define $f_n(z) = f(z/n)$, $n \geq 1$. Show that $\{f_n\}$ is a normal family on the punctured disc iff the singularity of $f(z)$ at $z = 0$ is removable or a pole.

5.4. Let $D \subset \mathbb{C}$ be a bounded domain. Suppose $f : D \to D$ is a holomorphic automorphism (conformal bijection). Let $f_n(z) = f \circ f \circ \cdots \circ f$ (n times).

(i) Prove that the sequence $\{f_n\}$ has a subsequence that converges either to a constant or to an automorphism of D.

(ii) If the whole sequence $\{f_n\}$ converges to g, then $f(z) \equiv z$.

5.5. (For 9056 only.) Prove that any conformal bijective map between two rectangles in $\mathbb{C}$ sending vertices to vertices is a linear map.