

1. Consider the sequence defined by the recursion $a_{n+1} = \sqrt{a_n}$.

(a) Show that if $0 < a_0 < 1$, then a_n is convergent.

Solution.

Observe that if $x \in (0, 1)$, then (i) $\sqrt{x} \in (0, 1)$ and (ii) $\sqrt{x} > x$. Thus if $a_n \in (0, 1)$ then $0 < a_n < a_{n+1} < 1$. By induction (formal proof not necessary), if our sequence begins with $a_0 \in (0, 1)$ it will be bounded and increasing. By the monotone sequence theorem it will be convergent.

(b) Given that a_n is convergent and $0 < a_0 < 1$, evaluate $L = \lim_{n \rightarrow \infty} a_n$.

Solution.

Since $f(x) = \sqrt{x}$ is continuous on its domain we have

$$\begin{aligned} L &= \lim_{n \rightarrow \infty} a_{n+1} \\ &= \lim_{n \rightarrow \infty} \sqrt{a_n} \\ &= \sqrt{\lim_{n \rightarrow \infty} a_n} \\ &= \sqrt{L} \end{aligned}$$

L therefore solves the equation $L = \sqrt{L}$, which is equivalent to $L^2 = L$ or $L(L-1) = 0$. There are two possibilities, namely $L = 0$ and $L = 1$. Since a_n is increasing we must have $L \geq a_0 > 0$, and therefore $L = 1$.

2. Evaluate the sum of the (convergent) series $\sum_{n=1}^{\infty} \frac{2}{n(n+1)}$.

Solution

Noting that $\frac{2}{n(n+1)} = \frac{2}{n} - \frac{2}{n+1}$ we find that the partial sums of this series

are

$$\begin{aligned} s_N &= \sum_{n=1}^N \frac{2}{n(n+1)} \\ &= \sum_{n=1}^N \left[\frac{2}{n} - \frac{2}{n+1} \right] \\ &= \sum_{n=1}^N \frac{2}{n} - \sum_{n=1}^N \frac{2}{n+1} \\ &= \left[\frac{2}{1} + \frac{2}{2} + \dots + \frac{2}{N-1} + \frac{2}{N} \right] - \left[\frac{2}{2} + \frac{2}{3} + \dots + \frac{2}{N} + \frac{2}{N+1} \right] \\ &= 2 - \frac{2}{N+1} \end{aligned}$$

Thus

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{2}{n(n+1)} &= \lim_{N \rightarrow \infty} s_N \\ &= \lim_{N \rightarrow \infty} \left(2 - \frac{2}{N+1} \right) \\ &= 2 \end{aligned}$$

3. Determine whether the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n+1}$ is absolutely convergent, conditionally convergent or divergent. Justify your answer.

Solution

The series is divergent since $\lim_{n \rightarrow \infty} (-1)^{n+1} \frac{n}{n+1}$ does not exist (even terms converge to -1 , odd terms to 1).

4. Let s denote the sum of the (convergent) infinite series $\sum_{n=1}^{\infty} \frac{1}{n^4}$.

- (a) It can be shown that $1 + \frac{1}{2^4} + \frac{1}{3^4} + \dots + \frac{1}{10^4} = 1.0820\dots$. Use this fact to derive an upper and lower bound for s .

Solution

If $R_n = \frac{1}{(n+1)^4} + \frac{1}{(n+2)^4} + \dots$ we know that

$$\int_{n+1}^{\infty} \frac{1}{x^4} dx \leq R_n \leq \int_n^{\infty} \frac{1}{x^4} dx.$$

And since $\int_n^{\infty} \frac{1}{x^4} dx = \frac{1}{3n^3}$ this yields the following

$$\frac{1}{3 \cdot 11^3} \leq R_{10} \leq \frac{1}{3 \cdot 10^3}.$$

Adding $s_{10} = 1.0820\dots$ to each side gives

$$1.0820\dots + \frac{1}{3 \cdot 11^3} \leq s \leq 1.0820\dots + \frac{1}{3 \cdot 10^3}.$$

- (b) In (a) we used 10 terms to estimate s . How many terms would we need to use in order to ensure that the resulting error was no larger than $\frac{10^{-6}}{3}$?

Solution Since $R_n \leq \int_n^{\infty} \frac{1}{x^4} dx = \frac{1}{3n^3}$, it suffices to ensure that $\frac{1}{3n^3} \leq \frac{10^{-6}}{3}$. This will occur if and only if $n \geq 100$, thus we need at least 100 terms.

5. Determine whether the series $\sum_{n=1}^{\infty} \frac{\sqrt{n} + 7}{\sqrt{n^3 + 3n - 1}}$ converges or diverges.

Solution

When n is large, we would expect that

$$\frac{\sqrt{n} + 7}{\sqrt{n^3 + 3n - 1}} \approx \frac{\sqrt{n}}{\sqrt{n^3}} = \frac{n^{1/2}}{n^{3/2}} = \frac{1}{n}.$$

To verify this let $b_n = \frac{1}{n}$ and observe that

$$\begin{aligned}
 \frac{a_n}{b_n} &= na_n \\
 &= \frac{n\sqrt{n} + 7n}{\sqrt{n^3 + 3n - 1}} \\
 &= \frac{n^{3/2} + 7n}{\sqrt{n^3 + 3n - 1}} \\
 &= \frac{n^{3/2}(1 + 7n^{-1/2})}{n^{3/2}\sqrt{1 + 3n^{-2} - n^{-3}}} \\
 &= \frac{1 + 7n^{-1/2}}{\sqrt{1 + 3n^{-2} - n^{-3}}}
 \end{aligned}$$

It is clear now that $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 1$, and since $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges, $\sum_{n=1}^{\infty} a_n$ also diverges by limit comparison.

6. Determine whether the series $\sum_{n=0}^{\infty} \frac{(-2)^n n!}{(2n)!}$ is absolutely convergent, conditionally convergent or divergent.

Solution

Let $a_n = \frac{(-2)^n n!}{(2n)!}$ so that $|a_n| = \frac{2^n n!}{(2n)!}$ and

$$\begin{aligned}
 \left| \frac{a_{n+1}}{a_n} \right| &= \frac{2^{n+1}}{2^n} \cdot \frac{(n+1)!}{n!} \cdot \frac{(2n)!}{(2(n+1))!} \\
 &= 2 \cdot (n+1) \cdot \frac{1}{(2n+2)(2n+1)} \\
 &= \frac{1}{2n+1}
 \end{aligned}$$

Thus $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 0 < 1$, and the series is absolutely convergent by the Ratio Test.

7. Determine whether the series $\sum_{n=2}^{\infty} (-1)^{n+1} \frac{1}{n \ln(n)}$ is absolutely convergent, conditionally convergent or divergent.

Solution

To check absolute convergence let $f(x) = \frac{1}{x \ln(x)}$ and observe that f is clearly positive and decreasing on $[2, \infty)$. Moreover the substitution $u = \ln(x)$ yields

$$\int_2^{\infty} \frac{1}{x \ln(x)} dx = \int_{\ln(2)}^{\infty} \frac{1}{u} du = \infty .$$

Therefore the series $\sum_{n=2}^{\infty} \frac{1}{n \ln(n)}$ diverges by the Integral Test, and our series is not absolutely convergent.

To check convergence let $b_n = \frac{1}{n \ln(n)}$ and observe that (i) b_n is clearly decreasing and (ii) $\lim_{n \rightarrow \infty} b_n = 0$. Therefore the series $\sum_{n=2}^{\infty} (-1)^{n+1} \frac{1}{n \ln(n)}$ converges by the Alternating Series Test.

Therefore the series $\sum_{n=2}^{\infty} (-1)^{n+1} \frac{1}{n \ln(n)}$ is conditionally convergent.

8. (a) Is the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\sqrt{n+1}}$ conditionally or absolutely convergent? Justify your answer.

Solution

The series is convergent by the Alternating Series Test. But $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}}$ clearly diverges, so this convergence is not absolute. Therefore the series is conditionally convergent.

- (b) How many terms would be required in order to estimate the sum of the series in part (a) with an error that does not exceed 10^{-4} ?

Solution

Since this series satisfies the conditions of the Alternating Series Test we know that

$$|R_n| \leq b_{n+1} = \frac{1}{\sqrt{n+2}}.$$

Thus in order to ensure that $|R_n| \leq 10^{-4}$ it suffices to ensure that $\frac{1}{\sqrt{n+2}} \leq 10^{-4}$, which requires $n \geq 10^8 - 2$. Thus if we use at least $10^8 - 2$ terms we can be sure the resulting error does not exceed 10^{-4} .

9. Determine the radius and interval of convergence for the power series

$$\sum_{n=0}^{\infty} \frac{(2x-7)^n}{3n+1}.$$

Solution

To begin note that $(2x-7)^n = 2^n \left(x - \frac{7}{2}\right)^n$, and we see that the series is centered at $a = \frac{7}{2}$. Now fix $x \neq \frac{7}{2}$ and let $a_n = \frac{(2x-7)^n}{3n+1}$, so that

$$\begin{aligned} \left| \frac{a_{n+1}}{a_n} \right| &= \frac{|2x-7|^{n+1}}{|2x-7|^n} \cdot \frac{3n+1}{3(n+1)+1} \\ &= |2x-7| \cdot \frac{3n+1}{3n+4} \end{aligned}$$

Thus

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = |2x-7| \cdot \lim_{n \rightarrow \infty} \frac{3n+1}{3n+4} = |2x-7|,$$

and the Ratio Test ensures that our series converges whenever $|2x-7| < 1$, or $|x - \frac{7}{2}| < \frac{1}{2}$, and diverges whenever $|x - \frac{7}{2}| > \frac{1}{2}$. Therefore the radius of convergence is $R = \frac{1}{2}$.

In order to determine the interval of convergence we must check the endpoints $\frac{7}{2} \pm \frac{1}{2}$, which are simply 3 and 4. When $x = 3$ the series becomes $\sum_{n=1}^{\infty} (-1)^n \frac{1}{3n+1}$, which converges by the Alternating Series Test. When $x = 4$ the series becomes $\sum_{n=1}^{\infty} \frac{1}{3n+1}$, which diverges by

comparison (limit or otherwise) with the harmonic series. Thus the interval of convergence is $[3, 4)$.

10. (a) Express $\frac{x}{1+x^4}$ as a power series. Be sure to indicate the radius and interval of convergence.

Solution

Using the geometric series (and assuming $|x| < 1$, so that $|-x^4| < 1$) we find that

$$\begin{aligned}\frac{1}{1+x^4} &= \frac{1}{1-(-x^4)} \\ &= \sum_{n=0}^{\infty} (-x^4)^n \\ &= \sum_{n=0}^{\infty} (-1)^n x^{4n} \\ &= 1 - x^4 + x^8 - x^{12} + x^{16} - \dots\end{aligned}$$

Thus

$$\begin{aligned}\frac{x}{1+x^4} &= x \sum_{n=0}^{\infty} (-1)^n x^{4n} \\ &= \sum_{n=0}^{\infty} (-1)^n x^{4n+1} \\ &= x - x^5 + x^9 - x^{13} + x^{17} - \dots\end{aligned}$$

The radius and interval of convergence are 1 and $(-1, 1)$.

- (b) Estimate $\int_0^1 \frac{x}{1+x^4} dx$ using the first three (non-zero) terms of an appropriate series.

Solution

Integrating term-by-term we obtain

$$\begin{aligned}\int_0^1 \frac{x}{1+x^4} dx &= \int_0^1 [x - x^5 + x^9 - x^{13} + x^{17} - \dots] dx \\ &= \int_0^1 x dx - \int_0^1 x^5 dx + \int_0^1 x^9 dx - \dots \\ &= \frac{1}{2} - \frac{1}{6} + \frac{1}{10} - \dots\end{aligned}$$

Thus an estimate based on the first three terms is simply

$$\int_0^1 \frac{x}{1+x^4} dx \approx \frac{1}{2} - \frac{1}{6} + \frac{1}{10} = \frac{13}{30}.$$

Note also that

$$\int \frac{x}{1+x^4} dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{4n+2},$$

whose interval of convergence is $(-1, 1]$, so that this term-by-term integration is in fact permitted.

11. Suppose that $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \frac{2}{3}$. Evaluate $\lim_{n \rightarrow \infty} a_n$.

Solution

Since $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \frac{2}{3}$, it follows that $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{2}{3}$ as well (the function $f(x) = |x|$ is continuous on all of \mathbb{R}). Since $\frac{2}{3} < 1$ the Ratio Test ensures that the series $\sum_{n=1}^{\infty} a_n$ converges, and therefore $\lim_{n \rightarrow \infty} a_n = 0$.