Calculus 1501B - Winter 2012
Assignment # 2

- Due Monday, February 13^{th} by the end of class.
- You may hand in your assignment at any time prior to the due date, either in my office (MC 268 - slip it under the door if I am not there) or in class.

1. Determine which of the following integrals converge.

(a) $\displaystyle\int_1^{\infty} \frac{\ln x}{x^2}\,dx$ (b) $\displaystyle\int_0^{1} \frac{1}{\sqrt{x(1-x)}}\,dx$

(c) $\displaystyle\int_{-\infty}^{\infty} \sqrt{|x|}\,dx$ (d) $\displaystyle\int_0^{\infty} \frac{\sin^2 x}{1+x^3}\,dx$

2. This problem concerns the improper integral

$$\int_0^1 \frac{1}{x^p}\,dx\ .$$

(a) For what values of p does the integral converge?

(b) When convergent, what is the value of the integral? Please note that I am requesting a formula for the value of this integral in terms of p.

(c) How does this set of values found in (a) compare with those for which $\int_1^{\infty} \frac{1}{x^p}\,dx$ converges? You may use the results of any examples done in class.

(d) There is a big difference in the "behaviour" of $\int_0^\infty \frac{1}{x^p}\,dx$ when $p = 1$ and $p \neq 1$. Describe this difference.

3. Determine whether or not the integral $\int_0^\infty x^{-1}e^{-x}\,dx$ converges. What does your answer tell you about the behaviour of $\Gamma(z)$ as $z \to 0^+$.

4. It can be shown that $\displaystyle\int_0^\infty \sqrt{x}e^{-x}\,dx = \sqrt{\pi}$. Use this fact to compute $\left(\frac{5}{2}\right)!$

5. Prove that $\displaystyle\lim_{n\to\infty} \frac{2n}{1-3n} = -\frac{2}{3}$.

6. Evaluate the following limits (you do not need to use the formal definition)

(a) $\displaystyle\lim_{n\to\infty} \sqrt[n]{2^n + 3^n}$

(b) $\displaystyle\lim_{n\to\infty} \left(\sqrt{n^2+n} - \sqrt{n^2+4n}\right)$

(c) $\displaystyle\lim_{n\to\infty} \frac{(-1)^n \sqrt{n}\sin(n^n)}{n+1}$

(d) $\displaystyle\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$

(e) $\displaystyle\lim_{n\to\infty} n\sin\left(\frac{1}{n}\right)$

Hints:

(a) $a^n + b^n = a^n\left[1 + \left(\frac{a}{b}\right)^n\right]$.

(b) Multiply by one.

(c) $\sin(n^n)$ looks intimidating, but don't be scared - it's actually not that big (or small).

7. Let $a_n = (-1)^n \frac{1}{n}$ and

$$f(x) = \begin{cases} 1+x & \text{if} \quad x \geq 0 \\ x & \text{if} \quad x < 0 \end{cases}$$

Is it true that $\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right)$? If not, does this violate Theorem 7 (page 695) in the textbook?

8. If $\lim_{n\to\infty} a_n = \infty$, then $\lim_{n\to\infty} \frac{1}{a_n} = 0$ (you do not need to prove this). Illustrate that the converse is not necessarily true by providing an example of a sequence for which $\lim_{n\to\infty} a_n = 0$ but $\lim_{n\to\infty} \frac{1}{a_n} \neq \infty$.

9. Suppose that $\lim_{n\to\infty} a_n = L$, and let $b_n = (-1)^n a_n$. For what values of L is b_n convergent?