Solutions to Practice Midterm 2

Problem 1. Evaluate the sum of the infinite series $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n^2+n}}$.

Solution. The n^{th} partial sum is

$$\begin{aligned}
s_n &= \sum_{k=1}^{n} \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n^2+n}} \\
&= \frac{\sqrt{2}-\sqrt{1}}{\sqrt{2}} + \frac{\sqrt{3}-\sqrt{2}}{\sqrt{6}} + \frac{\sqrt{4}-\sqrt{3}}{\sqrt{12}} + \cdots + \frac{\sqrt{n}-\sqrt{n-1}}{\sqrt{(n-1)^2+(n-1)}} + \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n^2+n}} \\
&= \left(1 - \frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}}\right) + \left(\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}}\right) + \cdots + \left(\frac{1}{\sqrt{n-1}} - \frac{1}{\sqrt{n}}\right) + \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right) \\
&= 1 - \frac{1}{\sqrt{n+1}}.
\end{aligned}$$

The sum of the infinite series is the limit of the sequence of partial sums, $\displaystyle\lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(1 - \frac{1}{\sqrt{n+1}}\right) = 1$.

Problem 2. For what values does the series $\displaystyle\sum_{n=0}^{\infty} \frac{4^n+6^n}{(2c)^n}$ converge?

Solution. When $|c| > 3$,

$$\begin{aligned}
\sum_{n=0}^{\infty} \left|\frac{4^n+6^n}{(2c)^n}\right| &= \sum_{n=0}^{\infty} \left|\frac{4^n}{(2c)^n} + \frac{6^n}{(2c)^n}\right| \\
&\le \sum_{n=0}^{\infty} \left|\frac{4^n}{(2c)^n}\right| + \sum_{n=0}^{\infty} \left|\frac{6^n}{(2c)^n}\right| = \sum_{n=0}^{\infty} \left|\frac{4}{(2c)}\right|^n + \sum_{n=0}^{\infty} \left|\frac{6}{(2c)}\right|^n,
\end{aligned}$$

which are both convergent geometric series. So, when $|c| > 3$, the series converges absolutely. When $|c| \le 3$ the sequence $\displaystyle\left(\frac{4^n+6^n}{(2c)^n}\right)_{n=0}^{\infty}$ does not converge to 0, so the series does not converge.

Problem 3. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{\cos\left(\frac{1}{n}\right)+n^4}{(n^2+n+1)(n^3+1)}$ converges or diverges.

Solution. Version 1: We have the following (in)equalities:

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{\cos\left(\frac{1}{n}\right) + n^4}{(n^2+n+1)(n^3+1)} &\geq \sum_{n=1}^{\infty} \frac{n^4-1}{(n^2+n+1)(n^3+1)} \\
&= \sum_{n=1}^{\infty} \frac{(n-1)\left(n^3+n^2+n+1\right)}{(n^2+n+1)(n^3+1)} \\
&\geq \sum_{n=1}^{\infty} \frac{(n-1)\left(n^3+n^2+n+1\right)}{(n^2+n+1)(n^3+n^2+n+1)} \\
&= \sum_{n=1}^{\infty} \frac{n-1}{n^2+n+1} \\
&\geq \sum_{n=1}^{\infty} \frac{n-1}{n^2+2n+1} = \sum_{n=1}^{\infty} \frac{n-1}{(n+1)(n+1)}.
\end{aligned}$$

Then, letting $u = x+1$, we can apply the integral test:

$$\begin{aligned}
\int_1^{\infty} \frac{x-1}{(x+1)(x+1)}\, dx &= \int_1^{\infty} \frac{u-2}{u^2}\, du \\
&= \int_1^{\infty} \frac{1}{u}\, du - \int_1^{\infty} \frac{2}{u^2}\, du.
\end{aligned}$$

The first integral diverges and the second is finite. Hence, this diverges by the integral test, so the original series also diverges.

Version 2: Use the limit comparison test against $\sum \frac{1}{n}$.

Problem 4. Suppose that $(a_n)_{n=1}^{\infty}$ is a positive sequence with the property that $\sum_{n=1}^{N} a_n \leq 2011 - \frac{1}{N}$ for all $N \geq 1$. Prove that $\sum_{n=1}^{\infty} a_n$ converges.

Solution. Let s_n denote the n^{th} partial sum of the series $\sum_{n=1}^{\infty} a_n$. Since $(a_n)_{n=1}^{\infty}$ is a positive sequence,

$$s_{k+1} = \sum_{n=1}^{k+1} a_n = \sum_{n=1}^{k} a_n + a_{k+1} \geq \sum_{n=1}^{k} a_n = s_k.$$

Therefore, the partial sums form a non-decreasing sequence. We are given that $s_n \leq 2011$ for all $n \in \mathbb{N}$, so the sequence of partial sums in bounded above. Then, by the monotone convergence theorem, the sequence of partial sums converges, which definitionally means that the series converges.

Problem 5. Determine whether the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+4}$ converges absolutely, converges conditionally, or diverges.

Solution. The sequence does not converge absolutely by the integral test. Let $u = x^3 + 4$, so $du = 3x^2 dx$.

$$\int_1^{\infty} \frac{x^2}{x^3+4}\, dx = \frac{1}{3} \int_1^{\infty} \frac{1}{u}\, du,$$

which diverges. The series does converge conditionally; the series is given as alternating, $\lim_{n\to\infty} \frac{n^2}{n^3+4} = 0$, and the sequence $\left(\frac{n^2}{n^3+4}\right)_{n=1}^{\infty}$ is decreasing in absolute values for all $n \geq 2$, since

$$\begin{aligned}\frac{n^2}{n^3+4} - \frac{(n+1)^2}{(n+1)^3+4} &= \frac{\Big((n+1)^3+4\Big)n^2 - (n+1)^2\left(n^3+4\right)}{\left(n^3+4\right)\Big((n+1)^3+4\Big)} \\ &= \frac{n^4+n^3+n^2-8n-5}{\left(n^3+4\right)\Big((n+1)^3+4\Big)} \geq 0\end{aligned}$$

whenever $n \geq 2$. Then, by the alternating series test, the series converges conditionally.

Problem 6. Suppose that $\sum_{n=1}^{\infty} a_n = \frac{2}{3}$. Evaluate $\lim_{n\to\infty} \frac{1}{1+a_n}$.

Solution. Since the series $\sum_{n=1}^{\infty} a_n$ converges, $\lim_{n\to\infty} a_n = 0$, for otherwise the series would diverge. Therefore, $\lim_{n\to\infty} \frac{1}{1+a_n} = \frac{1}{1+0} = 1$.

Problem 7. Determine whether the series $\sum_{n=1}^{\infty} (-1)^n \frac{n!}{n^n}$ is absolutely convergent, conditionally convergent, or divergent.

Solution. For all $n \in \mathbb{N}$,

$$\begin{aligned}\frac{n!}{n^n} &= \frac{\overbrace{n\,(n-1)\,(n-2)\cdot\ldots\cdot 3\cdot 2\cdot 1}^{\text{n terms}}}{\underbrace{n\cdot n\cdot n\cdot\ldots\cdot n\cdot n\cdot n}_{\text{n terms}}} \\ &= \left(\frac{n\,(n-1)\,(n-2)\cdot\ldots\cdot 5\cdot 4\cdot 3}{\underbrace{n\cdot n\cdot n\cdot\ldots\cdot n\cdot n\cdot n}_{\text{n-2 terms}}}\right)\left(\frac{2}{n^2}\right) \leq \frac{2}{n^2}.\end{aligned}$$

Then, $\sum_{n=1}^{\infty} \frac{n!}{n^n} \leq 2\sum_{n=1}^{\infty} \frac{1}{n^2}$, which converges, so the series converges absolutely.

Problem 8. Determine whether the series $\sum_{n=1}^{\infty} \frac{(-1)^n n}{(\ln n)^n}$ is absolutely convergent, conditionally convergent, or divergent.

Solution. Use the Root Test: since $\lim \sqrt[n]{n} = 1$, we have

$$\lim_{n\to\infty} \sqrt[n]{\frac{n}{(\ln n)^n}} = \lim_{n\to\infty} \frac{1}{\ln n} = 0,$$

and the series converges absolutely.

Problem 9. Determine the radius and interval of convergence for the power series $\sum_{n=1}^{\infty} \frac{(x+3)^n}{2^n\sqrt{n}}$.

Solution. The series converges whenever $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$.

$$\lim_{n\to\infty}\left|\frac{(x+3)^{n+1}\,2^n\sqrt{n}}{(x+3)^n\,2^{n+1}\sqrt{n+1}}\right| = \lim_{n\to\infty}\left|\left(\frac{x+3}{2}\right)\sqrt{\frac{n}{n+1}}\right|.$$

This is less than 1 whenever $|x+3| < 2$. When $x = -1$, the series is equal to $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$, which diverges, and when $x = -5$, the series is equal to $\sum_{n=1}^{\infty}(-1)^n\frac{1}{\sqrt{n}}$, which converges conditionally by the alternating series test. Therefore, the radius of convergence is 2, and the inteval of convergence is $[-5,-1)$.

Problem 10. Suppose that the series $\sum_{n=1}^{\infty} c_n (x-2)^n$ converges for $x = 4$ and diverges for all $x > 4$.

(a) Is it true that the series $\sum_{n=1}^{\infty} c_n (x-2)^{n-1}$ converges for $x = -2$?

(b) Is it true that the series $\sum_{n=1}^{\infty} nc_n (x-2)^{n-1}$

Solution. (a) Since the power series converges for $x = 4$ and diverges for all $x > 4$, the radius of convergence is 2, and for all $x \notin [0,4]$ the series diverges (it might also diverge at $x = 0$, but this doesn't matter). Since -2 is not within the interval of convergence, the series diverges for $x = -2$.

(b) The power series $\sum_{n=1}^{\infty} c_n (x-2)^n$ is differentiable and its derivative is $\sum_{n=1}^{\infty} nc_n (x-2)^{n-1}$. The derivative of a power series has the same radius of convergence as the original series, so $\sum_{n=1}^{\infty} nc_n (x-2)^{n-1}$ converges when $x = 3$.

Problem 11. (a) Express the function $f(x) = \frac{x}{(1-x)^2}$ as a power series and indicate the radius and interval of convergence.

(b) Use the answer from (a) to evaluate the sum of the series $\sum_{n=1}^{\infty}\frac{n5^n}{6^n}$.

Solution. (a)

$$\begin{aligned} f(x) = \frac{x}{(1-x)^2} &= \frac{-(1-x)+1}{(1-x)^2} \\ &= -\frac{1}{1-x} + \frac{1}{(1-x)^2} \\ &= -\sum_{n=0}^{\infty} x^n + \sum_{n=1}^{\infty} nx^{n-1} \\ &= \sum_{n=0}^{\infty} -x^n + (n+1)\,x^n = \sum_{n=0}^{\infty} nx^n. \end{aligned}$$

This series converges whenever $|x| < 1$, and diverges for both $x = 1$ and $x = -1$, since the sequences $(n)_{n=1}^{\infty}$ and $(n(-1)^n)_{n=1}^{\infty}$ both diverge. Hence, the radius of convergence is 1 and the inteval of convergence is $(-1,1)$.

(b) $\frac{5}{6}$ is within the interval of convergence of the series $\sum_{n=0}^{\infty} nx^n$. So,

$$\begin{aligned}\sum_{n=0}^{\infty} n\left(\frac{5}{6}\right)^n &= \frac{\left(\frac{5}{6}\right)}{\left(1-\left(\frac{5}{6}\right)\right)^2} \\ &= \frac{\left(\frac{5}{6}\right)}{\left(\frac{1}{36}\right)} = 36\left(\frac{5}{6}\right) = 30.\end{aligned}$$