THE NUMBER E.

These lecture notes are designed to provide supplementary material to Stewart, "Single Variable Calculus, Seventh Edition, with Early Transcendentals". This is far from a complete, or even rigorous treatment of set theory. Such a treatment would lead us too far astray. This is just enough to get us going in mathematics.

As this is an enriched course, some of the material taught is for the interest of the student and will not appear on exams. This matherial is differentiated from examinable material by a blue font colour.

1. The number e

These notes were put together in haste as the high school defintion of e is not sufficient. There are probably many typos and errors.

We will define e to be that number that satisfies

$$\lim_{h\to 0} \frac{e^h - 1}{h} = 1.$$

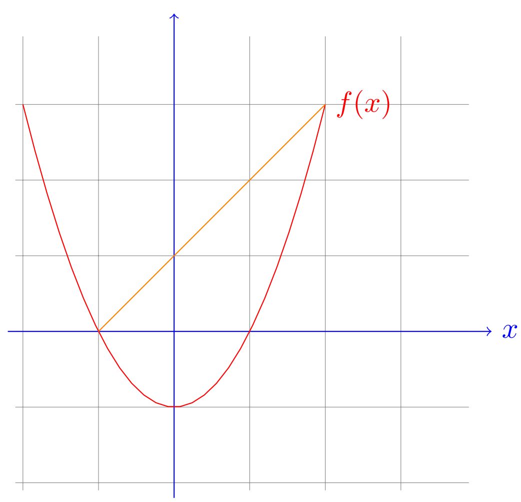
In order to make sense of this one needs to prove two things, that such a number exists and it is unique.

Here we will outline how one goes about proving this.

Definition 1.1. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be *strictly convex* if for every $t \in [0, 1]$ and every pair of real numbers $x_1 < x_2$ we have

$$f(tx_1 + (1-t)x_2) < tf(x_1) + (1-t)f(x_2).$$

In pictures it means that the graph of the function is below the secant from $(x_1, f(x_1))$ to $(x_2, f(x_2))$.



Proposition 1.2. *Suppose that $f(x)$ is a strictly convex function and $x_2' < x_2$ then for all $t, s \in [0,1]$ we have*

$$sf(x_1) + (1-s)f(x_2) < tf(x_1) + (1-t)f(x_2).$$

Proof. This is clear from the picture, but an actual proof will appear here later.
***************TODO************ □

Proposition 1.3. *Let $a > 0$ and $a \neq 1$ then the function $f(x) = a^x$ is a strictly convex function.*

Proof. This will follow from the precise definition of exponential from before.
***************TODO************ □

We will need to make use of the following result.

Theorem 1.4. *Let $f(x)$ be an increasing function defined on $(0, \infty)$. Suppose that f is bounded below, then*

$$\lim_{x \to 0^+} f(x)$$

exists.

Proof. Let

$$L = \inf\{f(x) | x \in (0, \infty)\}.$$

Note that L exists as the set has a lower bound. One easily shows that

$$L = \lim_{x \to 0^+} f(x)$$

***************TODO************ □

Proposition 1.5. *Suppose that $a \geq 1$. Then*

$$\lim_{x \to 0^+} \frac{a^x - 1}{x}$$

exists.

Proof. We take $f(x) = \frac{a^x - 1}{x}$. This function is bounded below by 0 on $(0, \infty)$. Further, this function is the slope of the secant from $(0, 1)$ to (x, a^x) on the graph of the exponential function, hence be 1.2 the function is increasing. The result follows from the above theorem. □

We consider the $A(x)$ function, with domain $[1, \infty)$, given by

$$A(x) = \lim_{h \to 0^+} \frac{x^h - 1}{h}.$$

This function makes sense by the above proposition.

Proposition 1.6. *We have*

1.1. *$A(x)$ is continuous*
1.2. *$A(1) = 0$*
1.3. *$A(x) > 1$ for x large enough.*

Proof. ***************TODO************ □

Corollary 1.7. *There is a number $e > 0$ with*

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1.$$

Proof. We apply the intermediate value theorem to $A(x)$. □