PROOF BY INDUCTION

These lecture notes are designed to provide supplementary material to Stewart, "Single Variable Calculus, Seventh Edition, with Early Transcendentals". This is far from a complete, or even rigorous treatment of set theory. Such a treatment would lead us too far astray. This is just enough to get us going in mathematics.

As this is an enriched course, some of the material taught is for the interest of the student and will not appear on exams. This matherial is differentiated from examinable material by a blue font colour.

1. Sigma Notation

Suppose that $f(i)$ is a function defined on the integers. Suppose that $m \leq n$ are both integers. The notation $\sum_{i=n}^{m} f(i)$ is shorthand for

$$f(m) + f(m+1) + \ldots + f(n-1) + f(n).$$

Example 1.1.

$$\sum_{i=4}^{7}(i^2 + i) = (4^2 + 4) + (5^2 + 5) + (6^2 + 6) + (7^2 + 7).$$

Example 1.2.

$$\sum_{n=1}^{100}\cos(n) = \cos(1) + \cos(2) + \ldots + \cos(100).$$

Example 1.3.

$$\sum_{n=k}^{l} e^n = e^k + e^{k+1} + \ldots + e^l.$$

Example 1.4. Evaluate the sum

$$\sum_{n=1}^{2011}(\frac{1}{n} - \frac{1}{n-1}).$$

We have

$$\begin{aligned}\sum_{n=1}^{2011}(\frac{1}{n} - \frac{1}{n+1}) &= (1/1 - 1/2) + (1/2 - 1/3) + (1/3 - 1/4) + \ldots (1/2011 - 1/2012) \\ &= 1 - 1/2012 \quad \text{cancelling terms.}\end{aligned}$$

2. Proof by induction

Suppose that we have a series of statement $P(n)$, one for each integer, $n \geq 1$. We would like to prove the validity. We can proceed as follows.
Step 1 : Prove $P(1)$.
Step 2 : Show that $P(n)$ implies $P(n+1)$, that is assume $P(n)$ is true and show that $P(n+1)$ follows from $P(n)$.

The reasoning behind this method of proof is that using steps 1 and 2 together we see that $P(2)$ must be true. Applying step 2 again we see that $P(3)$ must be true, and so on.

Example 2.1. Use induction to show that

$$1 + 2 + \ldots + n = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

So $P(n)$ is the assertion that

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

Step 1 : Substituting $n = 1$ into the above the left hand side works out to be 1. The right hand side works out to be

$$\frac{1(1+1)}{2} = 1.$$

So LHS=RHS and $P(1)$ is true.
Step 2 : So we assume that $P(n)$ is true that is

$$(*) \qquad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

We nee to prove $P(n+1)$, that is we need to prove

$$\sum_{i=1}^{n+1} i = \frac{(n+1)(n+2)}{2}$$

So starting with the LHS of $P(n+1)$ we have

$$\begin{aligned} 1 + 2 + \ldots + n + (n+1) &= (1 + 2 + \ldots + n) + (n+1) \\ &= \frac{n(n+1)}{2} + (n+1) \quad \text{by } * \\ &= \frac{n^2 + n + 2n + 2}{2} \\ &= \frac{(n+1)(n+2)}{2}. \end{aligned}$$

So we are done

3. Exercises

3.1. Prove the following identities by induction on

(a)
$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

(b)
$$\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

(c)
$$\sum_{i=0}^{n} 2^n = 2^{n+1} - 1$$

3.2. Use induction to show that when $x > 0$ we have

$$e^x > 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots + \frac{x^n}{n!}$$

3.3. Solve the following problems from appendix E:
Odd 1 to 33, 41, 43, 44, 45, 46.

3.4. Show that a set with n elements has distinct 2^n subsets. (Note that the empty set is a subset of every set.

3.5. Consider a sequence defined recursively by $a_1 = 1$ and $a_{n+1} = 3 - 1/a_n$. Show that $a_n < 3$ and $a_{n+1} > a_n$ for each n.