SETS

These lecture notes are designed to provide supplementary material to Stewart, "Single Variable Calculus, Sixth Edition, with Early Transcendentals". This is far from a complete, or even rigorous treatment of set theory. Such a treatment would lead us too far astray. This is just enough to get us going in mathematics.

As this is an enriched course, some of the material taught is for the interest of the student and will not appear on exams. This matherial is differentiated from examinable material by a blue font colour.

1. Sets

1.1. **Sets.** For our purposes a *set* is just a collection of distinct objects. When the set consists of a finite collection we can just list its contents inside curly braces to pin down which set we are talking about.

Example 1.1. The set of suits in a deck of cards is denoted by

$$\{\heartsuit, \spadesuit, \diamondsuit, \clubsuit\}$$

Example 1.2. The set of whole numbers between $1/2$ and 4.1 is denoted by

$$\{1, 2, 3, 4\}.$$

Example 1.3. The set with no elements $\{\}$ is called *the empty set* and is usually denoted $\emptyset$.

When the set consists of an infinite number of elements we cannot write down all the elements in a list. We need new notation to indicate its contents. Ellipsis, $\ldots$, indicate a continuation of a pattern. So the set of natural numbers (nonnegative whole numbers) can be denoted by

$$\{0, 1, 2, 3, 4, \ldots\}.$$

This set comes up so often in mathematics that it has a special symbol, $\mathbb{N}$. Other commonly used sets are the set of reals $\mathbb{R}$ and the set of complex numbers $\mathbb{C}$.

The objects in a set are called *elements*. We indicate that an x belongs to the set S by writing $x \in S$. If we are given two sets S and T and every element of S belongs to T we say that S is a *subset* of T. Symbolically we write

$$S \subseteq T$$

to indicate that S is a subset of T.

Example 1.4. We have

$$\{1, 2\} \subseteq \{1, 2, 3, 4\}.$$

Example 1.5. The emptyset $\{\}$ is a subset of every set.

Example 1.6. The natural numbers are a subset of the real numbers so $\mathbb{N} \subseteq \mathbb{R}$.

Suppose that S is a set and $C(s)$ is some condition on the elements of $s \in S$. We can form the subset of S consisting of those elements where the condition is true. Symbolically this denoted by

$$\{s \in S \mid C(s)\}.$$

This is commonly referred to as *set builder notation*.

Example 1.7. Lets say we want to form the collection of odd natural numbers. Notice that "s is odd" is a condition on the collection of natural numbers. Using the notation introduced above the collection of odd natural numbers can be constructed as

$$\{s \in \mathbb{N} \mid s \text{ is odd}\} = \{1, 3, 5, 7, \ldots\}.$$

Example 1.8. The interval in the real line consisting of numbers bigger than or equal to 0 but less than or equal to 1 is

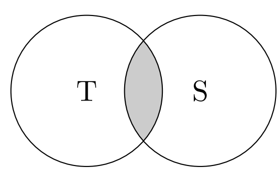
$$[0, 1] = \{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$$

Example 1.9. Suppose that we have real numbers $a < b$. Then we can form various intervals

$$\begin{aligned}
[a, b] &= \{x \in \mathbb{R} | a \leq x \leq b\} \\
[a, b) &= \{x \in \mathbb{R} | a \leq x < b\} \\
(a, b] &= \{x \in \mathbb{R} | a < x \leq b\} \\
(a, b) &= \{x \in \mathbb{R} | a < x < b\} \\
(a, \infty) &= \{x \in \mathbb{R} | a < x\} \\
[a, \infty) &= \{x \in \mathbb{R} | a \leq x\} \\
(-\infty, b) &= \{x \in \mathbb{R} | x < b\} \\
(-\infty, b] &= \{x \in \mathbb{R} | x \leq b\}
\end{aligned}$$

1.2. **Operations on Sets.** Two sets S and T are *equal* if they have the same elements, that is $x \in S$ if and only if $x \in T$. We denote equality of sets by writing $S = T$. Notice that $S = T$ is equivalent to asserting that $S \subseteq T$ and $T \subseteq S$. (why?)

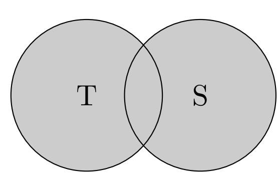
Consider two sets S and T. We can form a new set $S \cap T$ that consists of all elements that belong to both S and T. It is called the intersection of S and T. Pictorially it is the shaded region below :



Example 1.10.

$$\{1, 2, 3, 4\} \cap \{x \in \mathbb{R} | 0 \leq x \leq 1\} = \{1\}$$

We can also form the set $S \cup T$ consisting of elements that belong to at least one of S or T. This set is called the *union* of S and T. Pictorially it is the shaded region below :



Example 1.11.

$$\{1\} \cup \{2\} = \{1, 2\}$$

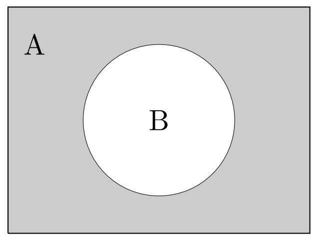
Example 1.12.

$$\{1,2,3,4\} \cup \{x \in \mathbb{R} | 0 \leq x \leq 1\} = \{x \in \mathbb{R} 0 \leq x \leq 1 \text{ or } x = 2,3,4\}$$

Suppose that B is a subset of A, that is $B \subseteq A$. The *complement* of B in A is

$$\{a \in A \mid a \notin B\}.$$

The complement is denoted symbolically as $A \setminus B$. In pictures :



Example 1.13.

$$\{1,2\} \setminus \{1\} = \{2\}$$

Example 1.14.

$$[0,1] \setminus \{0,1\} = (0,1) = \{x \in \mathbb{R} | 0 < x < 1\}$$

Suppose that S and T are sets. We can form the set of pairs

$$S \times T = \{(s,t) \mid s \in S,\ t \in T\}.$$

The set is called the *cartesian product* of S and T.

Example 1.15. Suppose that $S = \{1,4\}$ and $T = \{\alpha, \bullet\}$ then

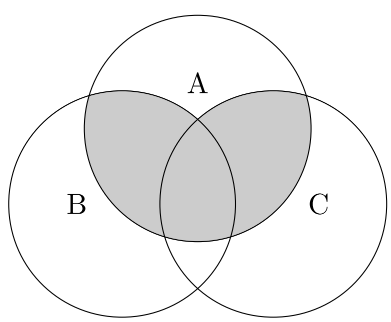
$$S \times T = \{(1,\alpha),(2,\alpha),(1,\bullet),(2,\bullet)\}.$$

1.3. **Proving equality of sets.** In order to show that two sets S and T are equal we typically proceed by showing that $S \subseteq T$ and $T \subseteq S$.

Example 1.16. Consider three sets A, B and C. Show that

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

Pictorially :



Before proceeding the picture is not considered a mathematically rigorous solution to this problem. In other words, **drawing the above picture is not a complete solution to the question being posed.** The picture may guide you in formulating your answer. So how does one show that

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)?$$

As stated earlier we need to show that

$$A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$$

and

$$A \cap (B \cup C) \supseteq (A \cap B) \cup (A \cap C),$$

that is we have to prove two things.

Lets begin with

$$A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C).$$

To prove this we need to show that every element of $A \cap (B \cup C)$ belongs to $(A \cap B) \cup (A \cap C)$. So we begin with a typical element $x \in A \cap (B \cup C)$. Unwinding the definition of intersection $(\cap)$ we see that $x \in A$ and $x \in B \cup C$. Now unwinding the definition of union we see that $x \in A$ and $(x \in B$ or $x \in C)$. Hence $x \in A \cap B$ or $x \in A \cap C$, using the definition of intersection. Finally, using the definition of union we see that $x \in (A \cap B) \cup (A \cap C)$. One can summarise this long winded discussion symbolically,

$$\begin{aligned} x \in A \cap (B \cup C) &\implies x \in A \text{ and } x \in B \cup C \\ &\implies x \in A \text{ and } (x \in B \text{ or } x \in C) \\ &\implies (x \in A \text{ and } x \in B) \text{ or } (x \in A \text{ and } x \in C) \\ &\implies x \in A \cap B \text{ or } x \in A \cap C \\ &\implies x \in (A \cap B) \cup (A \cap C). \end{aligned}$$

The symbol $\implies$ should be read as "implies". The conclusion is that

$$A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C).$$

It remains to show that

$$A \cap (B \cup C) \supseteq (A \cap B) \cup (A \cap C).$$

This is done by reversing the above argument :

$$\begin{aligned} x \in (A \cap B) \cup (A \cap C) &\implies x \in A \cap B \text{ or } x \in A \cap C \\ &\implies (x \in A \text{ and } x \in B) \text{ or } (x \in A \text{ and } x \in C) \\ &\implies x \in A \text{ and } (x \in B \text{ or } x \in C) \\ &\implies x \in A \text{ and } x \in B \cup C \\ &\implies x \in A \cap (B \cup C), \end{aligned}$$

Hence

$$A \cap (B \cup C) \supseteq (A \cap B) \cup (A \cap C).$$

1.4. **Functions.**

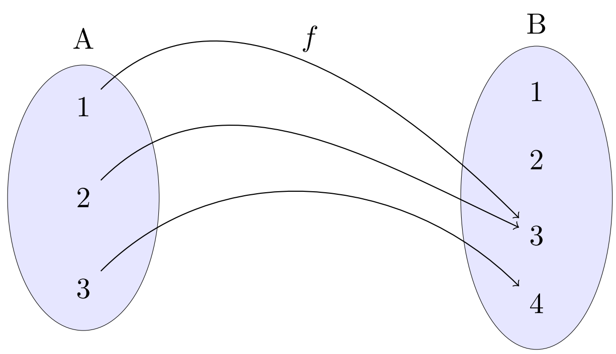
Definition 1.1. Let D and C be two sets. A *function from D to C* denoted $f : D \to C$ is a rule that assigns to each element of D and element of C. The set D is called the *domain* of the function. The set C is called the *codomain* of the function. (Note : C is not in general the range.)

This course is mostly concerned with functions $f : D \to \mathbb{R}$ where D is a subset of $\mathbb{R}$.

Example 1.17. Consider the sets $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4\}$. We define a function

$$f : A \to B$$

with $f(1) = 3,\ f(2) = 3,\ f(3) = 3.$ Pictorially :



2. Subsets of the Real Numbers

Let a and b be real numbers with $a \le b$. These two real numbers determine various subsets, called *intervals* :

$$\begin{aligned}
(a, b) &= \{x \in \mathbb{R} \mid a < x < b\} \\
(a, b] &= \{x \in \mathbb{R} \mid a < x \le b\} \\
[a, b) &= \{x \in \mathbb{R} \mid a \le x < b\} \\
[a, b] &= \{x \in \mathbb{R} \mid a \le x \le b\}
\end{aligned}$$

Definition 2.1. Let A be a subset of $\mathbb{R}$. A real number x is said to be an *upper bound* for A if $a \le x$ for every $a \in A$. Similarly a *lower bound* for A is a real number y such that $y \le a$ for every $a \in A$.

Example 2.1. Consider the subset $A = (1, 2) \cup (5, 7]$. The numbers $-1, 0, 1$ are all lower bounds. The numbers $10, 9$ and 7 are all upper bounds. The number 2 is not an upper bound.

Definition 2.2. Let A be a subset of $\mathbb{R}$. A real number x is said to be a *least upper bound* or *supremum* if x is an upper bound and if y is another upper bound then $x \le y$. We write $\sup A$ for the least upper bound.

Similarly a real number x is said to be a *greatest lower bound* or *infimum* if x is a lower bound and if y is another lower bound then $x \ge y$. We write $\inf A$ for the greatest lower bound.

Example 2.2. Consider the subset $A = (1, 2) \cup (5, 7]$.Then $\inf A = 1$ and $\sup A = 7$.

Example 2.3. A set need not have a supremum. For example

$$\{x \mid x > 0\} = (0, \infty)$$

has no upper bounds. We write $\sup(0, \infty) = \infty$ in this case. Similarly we have

$$\inf(-\infty, 4) = -\infty.$$

3. Properties of the set of Real Numbers

The set of real numbers, $\mathbb{R}$, has two binary operations, addition $+$ and multiplication $\cdot$ defined on it. These operations are subject to the usual rules,

3.1. $(a + b) + c = a + (b + c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
3.2. $a + b = b + a$ and $a \cdot b = b \cdot a$.
3.3. $a + 0 = a$ and $a \cdot 1 = a$.
3.4. $a + (-a) = 0$ for some $-a \in \mathbb{R}$.
3.5. If $a \neq 0$ then $a \cdot a^{-1} = 1$ for some $a^{-1} \in \mathbb{R}$
3.6. $a \cdot (b + c) = a \cdot b + a \cdot c$
3.7. If $x \geq y$ then $x + z \geq y + z$
3.8. If $x \geq 0$ and $y \geq 0$ then $xy \geq 0$
3.9. If a subset S of $\mathbb{R}$ has an upper bound then it has a supremum.

A consequence of this fact is that every subset T of $\mathbb{R}$ that has a lower bound has an infinum.

Proof. Let b a lower bound for T. Consider the set

$$S = -T = \{-t | t \in T\}.$$

As $b > t$ for all $t \in T$ we have that $-b < s = -t$ for all $s \in S$ by exercise 5 below. Hence the set has an upper bound and by this property, it has a supremum, that is a least upper bound. Lets call this supremum l. So $l > s = -t$, for all $t \in T$. Once again by exercise 5, we have that $-l < t$ for all $t \in T$. Hence $-l$ is a lower bound for S. Let k be another lower bound for T. Then arguing as above $-k$ is an upper bound for $S = -T$. So $-k \leq l$ as l is the lease upper bound (supremum) for $-T$. It follows that $k \leq -l$ and hece $-l$ is the greatest lower bound, that is the infinum. □

4. Exercises

4.1. Describe the following sets in interval notation :

$$\{x \in \mathbb{R} | 1 < x\}$$

$$\{x \in \mathbb{R} | 1 \leq x < 5 \text{ or } x < -1\}$$

4.2. Find the sup and inf of the following sets, when they exist

$$\{x \in \mathbb{R} | x < -7\}$$

$$\{x \in \mathbb{R} | 1 \leq x < 5 \text{ or } x < -1\}$$

$$(-3, 5) \cap [2, 10)$$

$$\{x \in \mathbb{R} | \pi < x < 12\} \cup [20, 100]$$

Deduce the following from the properties of the real numbers described in section 3

4.3. If $a < b$ then $-b < -a$.
4.4. If $a < b$ and $c > 0$ then $ac < bc$.

4.5. If $a < b$ and $c < 0$ then $ac > bc$.

4.6. * Show that the following equation has a solution in $\mathbb{R}$,

$$x^2 = 10$$

4.7. Suppose that A and B are subsets of C. Draw a picture to show that

$$(C \setminus A) \cup (C \setminus B) = C \setminus (A \cap B).$$

4.8. Now show that

$$(C \setminus A) \cup (C \setminus B) = C \setminus (A \cap B).$$

(**Note :** Your picture in the previous question is not a complete answer to this question.

4.9. Show that

$$(C \setminus A) \cap (C \setminus B) = C \setminus (A \cup B).$$

4.10. How many functions are there from $A = \{*\}$ to $B = \{f, g, h\}$.

4.11. How many functions are there from B to A, where A and B are as in the above question?

4.12. How many functions are there from a set with 2 elements to a set with 3 elements?

4.13. Let S and T be subsets of $\mathbb{R}$. Suppose that $S \subset T$ and $\sup T$ exists. Show that $\sup S$ exists and $\sup T \geq \sup S$.

4.14. How many functions are there from a set with n elements to a set with m elements?