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CALC 1501 LECTURE NOTES

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1. Mean Value Theorem

1.1. **Review: limit, continuity, differentiability.** We denote by $\mathbb{R}$ the set of real numbers. A *domain* D of $\mathbb{R}$ is any subset of $\mathbb{R}$. Typically this will be on *open* interval (a, b) or a *closed* interval $[a, b]$. A function of a real variable is a function $f : D \to \mathbb{R}$, where D is a domain of $\mathbb{R}$.

Definition 1.1 (The $\epsilon - \delta$ Definition)**.** *We say that a function $f(x)$ has a limit L as x approaches a point x_0 and write $\lim\limits_{x\to x_0} f(x) = L$, if for any $\epsilon > 0$ there exists $\delta > 0$ such that whenever $0 < |x - x_0| < \delta$ (and $x \in D$) we have $|f(x) - L| < \epsilon$.*

The meaning of the above definition is that by choosing a sufficiently small interval $(x_0-\delta, x_0+\delta)$ of the point x_0 we can ensure that the values of $f(x)$ on this interval (excluding x_0) do not deviate from L by more than ϵ.

Example 1.1. We will use this definition to prove that $\lim_{x\to 0} \frac{\sin x}{x} = 1$. For this we need to show that for any $\epsilon > 0$, there exists a choice of $\delta > 0$ such that

$$\left|\frac{\sin x}{x} - 1\right| < \epsilon, \quad \text{whenever} \quad |x| < \delta.$$

First we recall the following inequality from trigonometry: for $0 < x < \pi/2$,

$$\sin x < x < \tan x. \tag{1}$$

If we divide $\sin x$ by the three terms in the above inequality we obtain

$$\frac{\sin x}{\sin x} > \frac{\sin x}{x} > \frac{\sin x}{\tan x} \quad \Rightarrow \quad 1 > \frac{\sin x}{x} > \cos x.$$

From this we conclude that

$$0 < 1 - \frac{\sin x}{x} < 1 - \cos x = 2\sin^2\frac{x}{2} < 2\sin\frac{x}{2} < x,$$

where in the last step we again used inequality (1). It follows that

$$\left|\frac{\sin x}{x} - 1\right| < |x|.$$

So, given $\epsilon > 0$, we can take $\delta = \epsilon$ to satisfy the definition of limit. $\diamond$

Definition 1.2. *We say that a function $f : D \to \mathbb{R}$ is continuous at a point $x_0 \in D$ if*

$$\lim_{x\to x_0} f(x) = f(x_0). \tag{2}$$

Using the $\epsilon - \delta$ definition this can be stated as follows: given $\epsilon > 0$, there exists $\delta > 0$ such that whenever $|x - x_0| < \delta$ we have $|f(x) - f(x_0)| < \epsilon$.

Example 1.2. Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = x$. Let x_0 be any real number. Then $f(x)$ is continuous at x_0. Indeed, using the $\epsilon - \delta$ definition we have $|f(x) - f(x_0)| = |x - x_0| < \epsilon$. This inequality can be ensured by taking $\delta = \epsilon$. $\diamond$

Theorem 1.3. *If f and g are continuous functions on a domain D, then so are the functions $f+g$, $f\cdot g$, and $c\cdot f$, where c is any constant. The function f/g is continuous at all points of D where $g\neq 0$. Further, if g is a function defined on the range of f, then the function $g\circ f = g(f(x))$ is continuous on D.*

Using the above theorem and the fact that $f(x) = x$ is a continuous function as shown in Example 1.2, we conclude that any polynomial is a continuous function, and any rational function (the quotient of two polynomials) is continuous at all points where the denominator does not vanish.

Example 1.3. Let $f(x) = \sqrt{x}$. We will use the $\epsilon - \delta$ definition to show that this function is continuous at any point $x_0 > 0$. Observer that

$$|\sqrt{x}-\sqrt{x_0}| = \frac{(|\sqrt{x}-\sqrt{x_0})(\sqrt{x}+\sqrt{x_0})|}{\sqrt{x}+\sqrt{x_0}} = \frac{|x-x_0|}{\sqrt{x}+\sqrt{x_0}} < \frac{|x-x_0|}{\sqrt{x_0}}$$

Now, let $\epsilon > 0$ be arbitrary. We choose $\delta = \epsilon\sqrt{x_0}$ (x_0 is a fixed number!). Then

$$|\sqrt{x}-\sqrt{x_0}| < \frac{|x-x_0|}{\sqrt{x_0}} < \frac{\epsilon\sqrt{x_0}}{\sqrt{x_0}} = \epsilon,$$

which proves the continuity. $\diamond$

Example 1.4. Let

$$f(x) = \begin{cases} 0, & x\neq 0, \\ 1, & x = 0. \end{cases}$$

Then $\lim_{x\to 0} f(x)$ exists and equals zero, but it differs from the value of f at the origin since $f(0) = 1$. Therefore, equation (2) does not hold, and $f(x)$ is not continuous at the origin. However, letting $f(0) = 0$ will make this function continuous everywhere. $\diamond$

Example 1.5 (Dirichlet's function)**.** Recall that a *rational number* is the quotient of two integers. The set of all rational numbers is denoted by $\mathbb{Q}$. All real numbers that are not rational are called *irrational*. They form a set $\mathbb{R}\setminus\mathbb{Q}$. Define

$$d(x) = \begin{cases} 1, & x\in\mathbb{Q}, \\ 0, & x\in\mathbb{R}\setminus\mathbb{Q}. \end{cases}$$

This function is discontinuous at all points. Indeed, let x_0 be any real number, say x_0 is rational. Then for any $\delta > 0$, the interval $(x_0-\delta, x_0+\delta)$ necessarily contains an irrational number, and therefore, $|d(x)-d(x_0)| = 1$ for $|x-x_0| < \delta$. Thus, for $\epsilon < 1$, no choice of δ will satisfy the condition of Definition 1.2. A similar argument will work if x_0 is irrational. $\diamond$

Example 1.6. Let

$$f(x) = \begin{cases} \sin\frac{1}{x}, & x\neq 0, \\ 0, & x = 0. \end{cases}$$

The function f (see Fig. 1) is defined for all x. It is continuous for all $x\neq 0$ because it is a composition of a continuous functions $1/x$ and $\sin x$. But $f(x)$ does not have a limit as $x\to 0$ (why?), and therefore $f(x)$ is not continuous at the origin. Note that there is no choice of $f(0)$ that will make this function continuous at the origin. $\diamond$

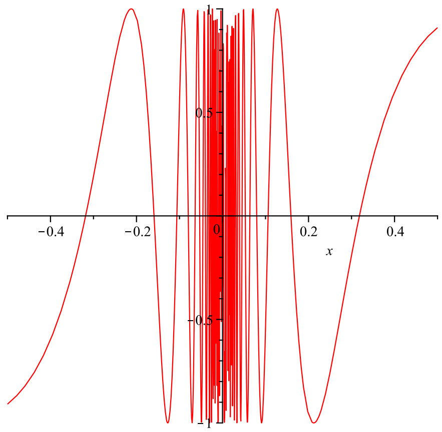

FIGURE 1. The graph of $\sin\frac{1}{x}$

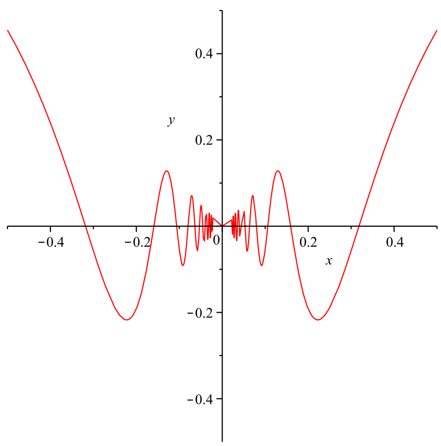

FIGURE 2. The graph of $x\sin\frac{1}{x}$

Example 1.7. Let

$$f(x) = \begin{cases} x\sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}.$$

This function (see Fig. 2) is continuous everywhere. To prove the continuity at the origin, let us verify the $\epsilon - \delta$ definition. We have

$$|f(x) - f(0)| = \left| x\sin\frac{1}{x} \right| < \epsilon.$$

Since $|x\sin\frac{1}{x}| < |x|$ for all $x \neq 0$, we have

$$|f(x) - f(0)| = \left| x\sin\frac{1}{x} \right| < |x| < \epsilon,$$

and so we may take $\delta = \epsilon$. Intuitively, $\lim_{x\to 0} x\sin\frac{1}{x} = 0$ because $\sin\frac{1}{x}$ is bounded between -1 and 1, whereas x approaches zero. ◇

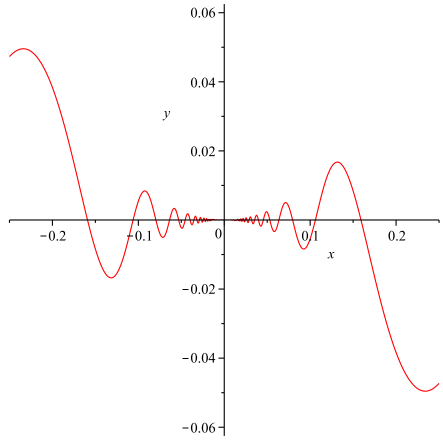

FIGURE 3. The graph of $x^2 \sin \frac{1}{x}$

Definition 1.4. *Let $f(x)$ be defined on an interval $D \subset \mathbb{R}$. Let $x_0 \in D$. We say that $f(x)$ is differentiable at x_0 if the limit*

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \tag{3}$$

exists. The value of the limit is defined to be $f'(x_0)$, the derivative of f at x_0.

Example 1.8. Let us apply the above definition to the function $f(x) = x^2$. The expression under the limit in equation (3) becomes

$$\frac{(x_0 + h)^2 - x_0^2}{h} = \frac{x_0^2 + 2x_0 h + h^2 - x_0^2}{h} = 2x_0 + h.$$

Clearly, the limit of the above expression equals $2x_0$, as $h \to 0$. Thus, we proved that $f(x) = x^2$ is differentiable at every point, and $(x^2)' = 2x$. $\diamond$

Example 1.9. Let

$$f(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}.$$

This function is continuous but not differentiable at the origin. The continuity was shown in Example 1.7. As for nondifferentiability, we have

$$\lim_{h \to 0} \frac{f(0 + h) - f(0)}{h} = \lim_{h \to 0} \frac{h \sin \frac{1}{h}}{h} = \lim_{h \to 0} \sin \frac{1}{h},$$

which does not exist. $\diamond$

Example 1.10. Let

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}.$$

This function (see Fig. 3) is continuous everywhere because it is the product of continuous functions x and $x \sin 1/x$ (as discussed in Example 1.7). To prove differentiability of this function at the origin let us compute the corresponding limit in (3).

$$\lim_{h \to 0} \frac{f(0 + h) - f(0)}{h} = \lim_{h \to 0} \frac{h^2 \sin \frac{1}{h}}{h} = \lim_{h \to 0} h \sin \frac{1}{h}.$$

As we saw in Example 1.7 this limit equals 0. Thus $f'(0) = 0$. $\diamond$

1.2. **Mean Value Theorem.**

Definition 1.5. *Suppose $f(x)$ is a function defined on a domain D. The function $f(x)$ is said to have an absolute (global) maximum at a point $c \in D$, if $f(c) \geq f(x)$ for all $x \in D$. The number $f(c)$ is called the absolute (global) maximum value of f on the domain D. The function f has an absolute (global) minimum at $c \in D$, if $f(c) \leq f(x)$ for all $x \in D$. The number $f(c)$ is called the absolute (global) minimum value of f on the domain D.*

Example 1.11. Consider a constant function $f(x) = c$, for some $c \in \mathbb{R}$. Then every point x is a global maximum and minimum of $f(x)$. On the other hand, the function $f(x) = x^3$ for $x \in \mathbb{R}$ does not attain a global maximum or minimum. The same is true if we consider this function on any *open* interval (a, b).

Theorem 1.6. *If $f(x)$ is continuous on a closed interval $[a, b]$, then $f(x)$ attains a maximum and a minimum value.*

The above theorem can be proved using the Axiom of Completeness for real numbers which will be stated when we discuss sequences.

Definition 1.7. *The function f defined on a domain D has a local maximum at a point $c \in D$, if there is an open interval $I \subset D$, such that $c \in I$, and $f(c) \geq f(x)$ for all $x \in I$. The function f has a local minimum at $c \in D$, if there is an open interval $I \subset D$, such that $c \in I$, and $f(c) \leq f(x)$ for all $x \in I$.*

Maxima and minima are called extreme points, or extrema.

Lemma 1.8. *Let $f(x)$ be a differentiable function on an interval (a, b). Suppose $x_0 \in (a, b)$. If $f'(x_0) > 0$, then for $x < x_0$ close to x_0 we have $f(x) < f(x_0)$, and $f(x) > f(x_0)$ for $x > x_0$ and close to x_0.*

The lemma above simply states that if $f'(x_0) > 0$, then $f(x)$ is an increasing function near x_0. A similar statement holds if we assume that $f'(x_0) < 0$ (see Exercise 1.4).

Proof. By definition,

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

If $f'(x_0) > 0$, then there exists a small interval $(x_0 - \delta, x_0 + \delta)$ such that

$$\frac{f(x) - f(x_0)}{x - x_0} > 0, \quad \text{for } x \neq x_0.$$

Suppose first that $x_0 < x < x_0 + \delta$. Then $x - x_0 > 0$, and from the above inequality we conclude that $f(x) - f(x_0) > 0$, or $f(x) > f(x_0)$. Now, if $x_0 - \delta < x < x_0$, then $x - x_0 < 0$, and the same inequality shows that $f(x) < f(x_0)$. $\square$

Theorem 1.9 (Fermat's Theorem)**.** [1] *Let $f(x)$ be defined on an interval $[a, b]$, and suppose that $f(x)$ attains a maximal (or minimal) value at a point $c \in (a, b)$. If $f(x)$ is differentiable at $x = c$, then $f'(c) = 0$.*

[1] This is a modern formulation of the theorem. It captures the essence of Fermat's method for finding maximal and minimal values of a function. The notion of derivative was not yet developed at Fermat's time.

Proof. We will assume that c is a maximum of $f(x)$, the case when c is a minimum can be treated in a similar way. Arguing by contradiction, suppose that $f'(c) \neq 0$. Then either $f'(c) > 0$ or $f'(c) < 0$. If $f'(c) > 0$, then Lemma 1.8 implies that $f(x) > f(c)$ for $x > c$ with x sufficiently close to c. Similarly, if $f'(c) < 0$, then $f(x) > f(c)$ for $x < c$. In both cases we see that $f(c)$ cannot be the maximum value of the function f. This contradiction proves the theorem. □

Geometrically, Fermat's theorem states that at extreme points the tangent line to the graph of the function f is horizontal, which should be intuitively clear. Also note, that if a maximal or a minimum value is attained at the end point of the interval $[a, b]$, then Fermat's theorem need not to hold.

Definition 1.10. *A point c is called a critical point of a differentiable function $f(x)$ if $f'(c) = 0$.*

Fermat's theorem now can be stated as follows: if c is a local maximum or minimum of a function $f(x)$, then c is a critical point of f. The converse to this statement is false: if $f'(c) = 0$, then it does not follow in general that c is a local maximum or a local minimum of $f(x)$. For example, if $f(x) = x^3$, then $f'(0) = 0$, but the origin is not an extreme point of x^3.

Theorem 1.11 (Rolle's Theorem)**.** [2] *Suppose $f(x)$ is continuous on the interval $[a, b]$, differentiable on (a, b), and $f(a) = f(b)$. Then there exists a number $c \in (a, b)$ such that $f'(c) = 0$.*

Proof. By Theorem 1.6, a continuous function on a closed interval $[a, b]$ attains its maximum value, say, M, and its minimum value, say, m. Consider two cases:

1. Suppose $M = m$. Then $f(x)$ on $[a, b]$ is a constant function, since $m \leq f(x) \leq M = m$ for all $x \in [a, b]$. Therefore, $f'(x) = 0$ for all x.

2. Suppose $M > m$. Since $f(a) = f(b)$, we know that either M or m is attained at some point c inside the interval (a, b), (i.e., not at the end points of the interval). In this case, it follows from Fermat's theorem that $f'(c)$ must be zero. □

Geometrically, Rolle's theorem states that if $f(a) = f(b)$, then there is a point c between a and b such that the tangent line to the graph of f at point c is horizontal. This occurs at a local maximum or a local minimum of $f(x)$.

Theorem 1.12 (Mean Value Theorem)**.** *Suppose that $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b). Then there exists a point $c \in (a, b)$ such that*

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

Proof. Define an auxiliary function

$$F(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a).$$

This function satisfies the conditions of Rolle's theorem. Indeed, it is continuous on $[a, b]$, because it is a difference of a continuous function $f(x)$ and a linear (hence continuous!) function

$$f(a) - \frac{f(b) - f(a)}{b - a}(x - a).$$

On the interval (a, b), we have

$$F'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}.$$

[2]Despite the name, Michel Rolle only suggested this result for polynomials in 1691.

Finally, $F(a) = f(a) - f(a) = 0$, and $F(b) = f(b) - f(a) - \frac{f(b)-f(a)}{b-a}(b-a) = f(b) - f(a) - (f(b) - f(a)) = 0$, and so $F(a) = F(b)$.

Therefore, we may apply Rolle's theorem to the function $F(x)$, and so there exists a point $c \in (a, b)$ such that $F'(c) = 0$. This means that

$$f'(c) - \frac{f(b) - f(a)}{b - a} = 0.$$

Hence,

$$f'(c) = \frac{f(b) - f(a)}{b - a},$$

which is exactly what we wanted to prove. □

Using the Mean Value Theorem we can now prove that only constant functions have everywhere vanishing derivatives.

Corollary 1.13. *Suppose $f(x)$ is a differentiable functions such that $f'(x) = 0$ for all x. Then $f(x)$ is a constant function.*

Proof. Choose any two points a and b in the domain of $f(x)$, say, $a < b$. By the Mean Value Theorem, there exists a point $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c) = 0.$$

It follows then that $f(b) = f(a)$. But this means that $f(x)$ is a constant function. □

1.3. **Proving inequalities.** The Mean Value Theorem can be used for proving inequalities.

Example 1.12. Prove that if $x > 0$, then

$$\ln(1 + x) < x.$$

Solution. Let $a = 0$, $b = x$, and $f(x) = \ln(1 + x) - x$. Then $f'(x) = \frac{1}{1+x} - 1 = -\frac{x}{1+x}$. By the Mean Value Theorem applied to the function f on the interval $[a, b] = [0, x]$, there exists a point $c \in (0, x)$ such that

$$f'(c) = \frac{f(x) - f(0)}{x - 0},$$

or

$$-\frac{c}{1 + c} = \frac{\ln(1 + x) - x}{x}. \tag{4}$$

Note that $c > 0$, and therefore, $-\frac{c}{1+c} < 0$. Therefore, equation (4) implies

$$\frac{\ln(1 + x) - x}{x} < 0.$$

Since $x > 0$, the numerator in the above inequality must be negative, i.e.,

$$\ln(1 + x) - x < 0,$$

which is what we had to prove. ⋄

Example 1.13. Prove that if $x > 0$, and $n > 1$, then

$$(1 + x)^n > 1 + nx.$$

Solution. Let $a = 0$, and $b = x$, and $f(x) = (1 + x)^n - (1 + nx)$. Then $f'(x) = n(1 + x)^{n-1} - n$, and by the Mean Value Theorem, we have

$$n(1 + c)^{n-1} - n = \frac{(1 + x)^n - (1 + nx) - 0}{x} \tag{5}$$

for some $c \in (0, x)$. Note that $1 + c > 1$, and for $n > 1$, we have $(1 + c)^{n-1} > 1$. Therefore,

$$n(1 + c)^{n-1} - n > 0.$$

From this and equation (5) we conclude that

$$\frac{(1 + x)^n - (1 + nx)}{x} > 0.$$

Since $x > 0$, this yields the desired inequality. ⋄

Exercises

1.1. Use a similar strategy as in Example 1.3 to show that the following functions are continuous on the specified domain:
  (a) $f(x) = 2x + 1$, for $x_0 \in \mathbb{R}$,
  (b) $f(x) = x^2$, for $x_0 \in \mathbb{R}$,
  (c) $f(x) = 1/x$ for $x_0 \neq 0$.

1.2. Prove, using the definition, that the function $f(x) = x^3$ is differentiable at all points.

1.3. Show that the function in Example 1.10 does not have the second order derivative at $x = 0$.

1.4. Formulate and prove a statement similar to Lemma 1.8 for the case when $f'(x_0) < 0$.

1.5. Give an example of a function which is defined on the closed interval $[0, 1]$ but is not bounded there.

1.6. Give an example of a function which is continuous on the interval $(-\infty, 0]$ but does not attain global maximum or minimum.

1.7. On the interval $(0, 1)$ find a point c such that the tangent line to the graph of the function $y = x^3$ at the point (c, c^3) is parallel to the straight line passing through the points $(0, 0)$ and $(1, 1)$.

1.8. Prove that if a nonconstant function $f(x)$ satisfies the conditions of Rolle's theorem on the interval $[a, b]$, then there exist points x_1 and x_2 on the interval (a, b) such that $f'(x_1) < 0$ and $f'(x_2) > 0$.

In the next problems prove the given inequality using the Mean Value Theorem.

1.9. $2\sqrt{x} > 3 - \dfrac{1}{x}$, for $x > 1$.

1.10. $\sin x < x$, for $x > 0$.

1.11. $\cos x > 1 - \dfrac{x^2}{2}$, for $x > 0$.

1.12. $\sin x > x - \dfrac{x^3}{6}$, for $x > 0$.

1.13. $\tan x > x$, for $0 < x < \frac{\pi}{2}$.

1.14. $e^x > 1 + x, \quad \text{for } x > 0.$

1.15. $e^x > 1 + x + \dfrac{x^2}{2}, \quad \text{for } x > 0.$

1.16. $e^x > 1 + x + \dfrac{x^2}{2} + \cdots + \dfrac{x^n}{n!}, \quad \text{for } x > 0.$ (Hint: use mathematical induction)