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CALC 1501 LECTURE NOTES

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4. Sequences

Definition 4.1. *A sequence s is a function $s : \mathbb{N} \to \mathbb{R}$. It can be thought of as a list of numbers*

$$s_1, s_2, s_3, \dots,$$

where $s_n = s(n)$ for $n \in \mathbb{N}$.

Example 4.1.

(i) $\{s_n\} = \left\{1, \frac{1}{2}, \frac{1}{3}, \dots\right\}$. The corresponding function $s : \mathbb{N} \to \mathbb{R}$ is given by $s(n) = \frac{1}{n}$.

(ii) Let

$$\{s_n\} = \left\{\frac{1}{2}, \frac{1}{6}, \frac{1}{12}, \frac{1}{20}, \dots\right\}.$$

Here $s(n) = \frac{1}{n \cdot (n+1)}$.

(iii) $\{s_n\} = \{1, -1, 1, -1, 1, -1 \dots\}$. For this sequence we can take, for example, $s_n = (-1)^{n+1}$.

◇

A sequence is defined *inductively* (or *recursively*) if $s_n = f(s_1, \dots s_{n-1})$, i.e., each term of the sequence is defined as a function of previously defined terms.

Example 4.2. (Fibonacci sequence[1].) By definition, the first two terms of the Fibonacci sequence $\{f_n\}$ are 1 and 1, and each consequent number is the sum of the previous two. Inductively this can be defined as follows.

$$f_1 = f_2 = 1, \quad f_n = f_{n-1} + f_{n-2}, \quad \text{for } n > 2.$$

The first several terms of the Fibonacci sequence can be easily computed to be

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \dots.$$

◇

Example 4.3. $s_1 = \sqrt{2}$, $s_n = \sqrt{2 + s_{n-1}}$ for $n > 1$. Then

$$s_1 = \sqrt{2}, s_2 = \sqrt{2+\sqrt{2}}, s_3 = \sqrt{2+\sqrt{2+\sqrt{2}}}, s_4 = \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}, \dots$$

◇

Definition 4.2. *A sequence $\{s_n\}$ converges to a real number L if for every positive number ϵ, there exists an $N \in \mathbb{N}$ such that $|s_n - L| < \epsilon$ for all $n > N$. In this case we write*

$$\lim_{n\to\infty} s_n = L.$$

If $\{s_n\}$ does not converge, it is said to diverge.

[1]The Fibonacci sequence is named after Leonardo of Pisa, who was known as Fibonacci (a contraction of filius Bonaccio, "son of Bonaccio"). Fibonacci's 1202 book *Liber Abaci* introduced the sequence to Western European mathematics, although the sequence had been previously described in Indian mathematics.

The above definition is sometimes called the ϵ-N definition of convergence of a sequence.

Example 4.4. $\lim_{n\to\infty} \dfrac{1}{\sqrt{n}} = 0.$

To prove this, set $s_n = \frac{1}{\sqrt{n}}$, and $L = 0$. We need to show that given any $\epsilon > 0$, there exists an index $N > 0$ such that $|s_n - L| = |1/\sqrt{n}| < \epsilon$ for $n > N$. The inequality $1/\sqrt{n} < \epsilon$ is equivalent to $n > 1/\epsilon^2$. By taking $N = \lceil 1/\epsilon^2 \rceil$, we ensure that if $n > N$, then $|1/\sqrt{n}| < \epsilon$. (Recall that $\lceil x \rceil$ is the *ceiling* function; it equals the smallest integer bigger than or equal to x.) $\diamond$

Using a similar argument one can show that $\lim_{n\to\infty} \dfrac{1}{n^p} = 0$ for $p > 0$ (Exercise 4.1(i)).

Example 4.5. $\lim_{n\to\infty} \dfrac{n+1}{n} = 1.$

Let $s_n = \frac{n+1}{n}$, and $L = 1$. Then

$$|s_n - L| = \left| \frac{n+1}{n} - 1 \right| = \left| \frac{1}{n} \right| < \epsilon,$$

and therefore, the choice of $N = \lceil 1/\epsilon \rceil$ will ensure that $|s_n - L| < \epsilon$. $\diamond$

Example 4.6. $\lim_{n\to\infty} \left(\dfrac{1}{2}\right)^n = 0.$

Set $s_n = \left(\frac{1}{2}\right)^n$, $L = 0$. Then

$$\left| \left(\frac{1}{2}\right)^n \right| < \epsilon \iff n \ln(1/2) < \ln \epsilon \iff n > \frac{\ln \epsilon}{\ln(1/2)}.$$

Note that $\ln(1/2) < 0$, and so division by this number reverses the inequality. We could take $N = \lceil \frac{\ln \epsilon}{\ln(1/2)} \rceil$, but then N becomes negative for $\epsilon > 1$. So a better choice is

$$N = \max \left\{ \left\lceil \frac{\ln \epsilon}{\ln(1/2)} \right\rceil, 1 \right\}.$$

$\diamond$

Definition 4.3. *We say that a sequence $\{s_n\}$ diverges to infinity, and write $\lim_{n\to\infty} s_n = \infty$ if for any real number M there exists an $N \in \mathbb{N}$ such that $s_n > M$ whenever $n \geq N$.*

Example 4.7. The Fibonacci sequence diverges to infinity. Indeed, starting with $n = 6$ we see that $f_n > n$. Therefore, given any number $M > 0$, $f_n > M$ for all $n > \max\{\lceil M \rceil, 6\}$. $\diamond$

Example 4.8. Investigate convergence of $\{r^n\}$ for different values of $r > 0$.

Suppose $r > 1$. Then if $M > 0$ is arbitrary, the inequality $r^n > M$ is satisfied for $n > \dfrac{\ln M}{\ln r}$. Thus r^n diverges to infinity if $r > 1$. If $r = 1$, then r^n is a constant sequence 1, hence converges to 1. Finally, if $0 < r < 1$, then $\lim_{n\to\infty} r^n = 0$. Indeed, given $\epsilon > 0$, for $n > \max \left\{ \dfrac{\ln \epsilon}{\ln r}, 1 \right\}$ the inequality $r^n < \epsilon$ holds. $\diamond$

The following theorem provides a convenient way of calculating the limit by reducing the problem to algebraic manipulation of existing limits. It can be proved directly using the ϵ-N definition of convergence.

Theorem 4.4 (Algebraic Limit Theorem)**.** *If* $\lim a_n = A$, $\lim b_n = B$, *then*

(i) $\lim(ca_n) = cA$ *for* $c \in \mathbb{R}$,

(ii) $\lim(a_n + b_n) = A + B$,

(iii) $\lim(a_n \cdot b_n) = A \cdot B$,

(iv) $\lim\left(\frac{a_n}{b_n}\right) = \frac{A}{B}$, *if* $b_n \neq 0$ *and* $B \neq 0$.

Example 4.9. Examples of use of the Algebraic Limit Theorem.

1. $\lim\limits_{n\to\infty} \frac{5n - 3n^2}{2n^2 + (-1)^n} = \lim\limits_{n\to\infty} \frac{5/n - 3}{2 + \frac{(-1)^n}{n^2}} = -\frac{3}{2}$. Here we use the result of Exercise 4.1(i) and also the fact that $\lim\limits_{n\to\infty} |a_n| = 0$ implies $\lim\limits_{n\to\infty} a_n = 0$, which follows directly from the ϵ-N definition of convergence.
2. $\lim\limits_{n\to\infty} \frac{n \ln n}{(n+1)^2} = \lim\limits_{n\to\infty} \frac{n}{n+1} \cdot \lim\limits_{n\to\infty} \frac{\ln n}{n+1} = 1 \cdot \lim\limits_{n\to\infty} \frac{1/n}{1} = 0$. Here we used l'Hôpital's rule.

$\diamond$

Further important reduction for computing limits of sequences is the following: if f is a continuous function and $\{s_n\}$ is a sequence that converges to limit L, then $\lim\limits_{n\to\infty} f(s_n) = f(L)$. In particular, this fact provides a useful trick for finding limits of sequences defined by an inductive formula: suppose $\{s_n\}$ is given by

$$s_{n+1} = f(s_n), \tag{1}$$

where f is a continuous function, and assume that the limit L of the sequence $\{s_n\}$ exists. Then, since $\lim s_n = \lim s_{n+1} = L$, we may take the limit on both sides of (1): $L = f(L)$. This will give the value of L, provided that the equation can be solved for L.

Example 4.10. Let $\{s_n\}$ be defined inductively by $s_1 = 1$, and

$$s_{n+1} = \frac{2s_n + 3}{4} \tag{2}$$

Assume that the limit of $\{s_n\}$ exists, say, $\lim\limits_{n\to\infty} s_n = L$. Then we can take the limit as $n \to \infty$ on both sides of (2). We get

$$\lim_{n\to\infty} s_{n+1} = \lim_{n\to\infty} \frac{2s_n + 3}{4}.$$

it follows that

$$L = \frac{2L + 3}{4}, \quad \text{so} \quad L = 3/2.$$

Hence, $\lim s_n = 3/2$. $\diamond$

If *apriori* it is not known that the limit of $\{s_n\}$ exists, then the calculation of L from equation (1) may produce unpredictable results, see Exercise 4.4 for details. Thus, justification of existence of the limit becomes an important problem in its own right.

Theorem 4.5 (Squeeze Theorem)**.** *If* $a_n \leq b_n \leq c_n$ *for* $n > n_0$ *and*

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L,$$

then $\lim\limits_{n\to\infty} b_n = L$.

Proof. Take any $\epsilon > 0$. We need to find $N > 0$ such that $|b_n - L| < \epsilon$ whenever $n > N$. Since $a_n \to L$, there is $N_1 > 0$ such that for $n > N_1$ we have $|a_n - L| < \epsilon$. An equivalent form of this inequality is

$$L - \epsilon < a_n < L + \epsilon.$$

Similarly, since $c_n \to L$, there is $N_2 > 0$ such that $|c_n - L| < \epsilon$ for $n > N_2$, or

$$L - \epsilon < c_n < L + \epsilon.$$

Take $N = \max\{N_1, N_2\}$. Then for $n > N$ we have

$$L - \epsilon < a_n \le b_n \le c_n < L + \epsilon,$$

which implies that $|b_n - L| < \epsilon$. □

Example 4.11. $\lim_{n\to\infty} \frac{5^n}{n^n} = 0$.
To prove this we use the Squeeze Theorem. Indeed, for $n > 6$,

$$0 < \frac{5^n}{n^n} = \left(\frac{5}{n}\right)^n < \left(\frac{5}{6}\right)^n.$$

We may take $a_n = 0$, $b_n = \frac{5^n}{n^n}$, and $c_n = \left(\frac{5}{6}\right)^n$. Since $\lim_{n\to\infty} \left(\frac{5}{6}\right)^n = 0$ by Example 4.8, the Squeeze Theorem implies that $\lim_{n\to\infty} \frac{5^n}{n^n} = 0$. ⋄

Definition 4.6. *A sequence $\{s_n\}$ is called increasing if $s_{n+1} \ge s_n$ for all n, strictly increasing if $s_{n+1} > s_n$ for all n. Decreasing and strictly decreasing sequences are defined similarly. Decreasing and increasing sequences are called* monotone *sequences.*

Example 4.12. $\left\{\frac{n}{n^2+1}\right\}$ is a decreasing sequence. This can be proved either by verifying the inequality $\frac{n+1}{(n+1)^2+1} \ge \frac{n}{n^2+1}$ for all n, or by showing that that function $f(x) = \frac{x}{x^2+1}$ has a negative derivative for $x > 1$. ⋄

Definition 4.7. *An upper bound of a non-empty subset S of $\mathbb{R}$ is a number b such that $b \ge s$, for any $s \in S$. A number l is a least upper bound or supremum of S, denoted by* $\sup S$, *if l is an upper bound of S, and if b is another upper bound of S then $l \le b$.*

Example 4.13.

(1) $S_1 = \{0, 1/2, 2/3, 3/4, \dots\}$. Then $\sup S_1 = 1$.
(2) $S_2 = \mathbb{N}$. This set is unbounded, and therefore, the upper bound for this set does not exist.
(3) Let

$$S_3 = \{\sin n,\ n \in \mathbb{N}\} = \{\sin 1, \sin 2, \sin 3, \dots\}.$$

This set is bounded above by 1, since $\sin x \le 1$ for any x. But is there $\sup S_3$? If n could attain any real value, then since $\sin(\frac{\pi}{2} + 2\pi k) = 1$, the supremum would be 1. However, since $n \in \mathbb{N}$, $\sin n \ne 1$ for any n. Therefore, if $\sup S_3$ exists, in order to find it, one needs to investigate how close a natural number n can come to the set of numbers of the form $\frac{\pi}{2} + 2\pi k, k \in \mathbb{N}$.

⋄

Lower bound and greatest lower bound (*infimum*) are defined similarly.

Definition 4.8. *A sequence $\{s_n\}$ is bounded above (resp. below) if the set*

$$\{s_n; n \in \mathbb{N}\} = \{s_1, s_2, s_3, \dots\}$$

has an upper (resp. lower) bound.

Axiom of Completeness. *Every nonempty set of real numbers that has an upper bound, has a least upper bound.*

The Axiom of Completeness distinguishes real numbers from rational numbers. For example, the set $S = \{x \in \mathbb{R} : x^2 < 2\}$ has a least upper bound $\sqrt{2}$. However, the set of rational numbers r, such that $r^2 < 2$, is bounded, but it does not have a least upper bound in $\mathbb{Q}$ ($\sqrt{2}$ is not rational!). Thus, the Axiom of completeness is false for rationals.

Let us return to Example 4.13(3). Since the set S_3 is bounded above by 1, the Axiom of Completeness guarantees that S_3 has a supremum, although it is a non-trivial problem to determine its value.

Theorem 4.9 (Monotone Convergence Theorem)**.** *Every bounded monotone sequence converges.*

Proof. Consider the case when $\{s_n\}$ is an increasing sequence bounded above. Since the set $S = \{s_n;\ n \in \mathbb{N}\}$ is bounded, by the Axiom of Completeness, there exists $l = \sup S$. We claim that l is the limit of $\{s_n\}$. Indeed, take any $\epsilon > 0$. Then since l is the supremum of S, there exits an index N such that $s_N > l - \epsilon$. But since the sequence is increasing, we have $s_n > l - \epsilon$ for all $n > N$. This means that $|l - s_n| < \epsilon$ for $n > N$, which proves that $\lim s_n = l$.

The case when $\{s_n\}$ is decreasing and bounded below can be proved in a similar way. □

Example 4.14. Consider the sequence defined in Example 4.3. We may use induction to show that $s_n < 2$ for all n. Indeed, $s_1 = \sqrt{2} < 2$. If $s_n < 2$, then $2 + s_n < 4$. Taking the square root on both sides, we get $\sqrt{2 + s_n} < 2$, which means that $s_{n+1} < 2$. This shows that the inequality $s_n < 2$ holds for all n.

Further, $\{s_n\}$ is increasing. Indeed, $s_n < \sqrt{2 + s_n}$ is equivalent to $s_n^2 - s_n - 2 < 0$, which holds true for $-1 < s_n < 2$. By the previous paragraph $s_n < 2$, and therefore, $s_n < s_{n+1}$ for all n.

Thus, $\{s_n\}$ is a bounded monotone sequence, and by the Monotone Convergence Theorem $\{s_n\}$ converges. The limit L can be found by taking the limit as $n \to \infty$ on both sides of $s_n = \sqrt{s_n + 2}$. We have

$$L = \sqrt{2 + L} \quad \Longrightarrow \quad L^2 - L - 2 = 0.$$

This equation has two roots: -1 and 2. Since $s_n > 0$ for all n, $L = 2$. ⋄

Example 4.15. Let a and b be two distinct positive real numbers with $a > b$. Recall that the *arithmetic mean* of a and b is the number $\frac{a+b}{2}$, the *geometric mean* equals $\sqrt{ab}$, and the *harmonic mean* is defined as $\frac{2ab}{a+b}$. It is easy to see that all three mean values are the numbers between b and a. We construct inductively the sequences of arithmetic and harmonic means as follows

$$\begin{aligned} a_1 &= \frac{a+b}{2}, \quad b_1 = \frac{2ab}{a+b} \\ a_2 &= \frac{a_1+b_1}{2}, \quad b_2 = \frac{2a_1b_1}{a_1+b_1} \\ &\dots\dots \\ a_{n+1} &= \frac{a_n+b_n}{2}, \quad b_{n+1} = \frac{2a_nb_n}{a_n+b_n} \\ &\dots\dots \end{aligned} \tag{3}$$

We have

$$(\sqrt{a}-\sqrt{b})^2 > 0 \implies \frac{a+b}{2} > \sqrt{ab} \implies \left(\frac{a+b}{2}\right)^2 > ab \implies \frac{a+b}{2} > \frac{2ab}{a+b}.$$

Thus the arithmetic mean of two numbers is always bigger than or equal to their harmonic mean. It follow from this calculation that

$$a_n > a_{n+1} > b_{n+1} > b_n$$

Therefore, the sequence $\{b_n\}$ is increasing and bounded above, while the sequence $\{a_n\}$ is decreasing and bounded below. It follows from the Monotone Convergence Theorem that both sequences converge. But what is the limit?

To answer this question assume that $\lim_{n\to\infty} a_n = \alpha$ and $\lim_{n\to\infty} b_n = \beta$. By taking the limit as $n \to \infty$ in the formula

$$a_{n+1} = \frac{a_n + b_n}{2}$$

we obtain the identity $\alpha = \frac{\alpha+\beta}{2}$, i.e., $\alpha = \beta$. To determine the value of α, observe that $a_1 \cdot b_1 = a \cdot b$. In fact, the identity

$$a_{n+1} \cdot b_{n+1} = a_n \cdot b_n = a \cdot b$$

holds for all n. By passing to the limit we obtain $\alpha \cdot \alpha = ab$, or $\alpha = \sqrt{ab}$, i.e., both sequences converge to the geometric mean of a and b. $\diamond$

Exercises

4.1. Using only Definition 4.2 prove
   (i) $\lim_{n\to\infty} \dfrac{1}{n^p} = 0, \ \ p > 0.$
   (ii) $\lim_{n\to\infty} \dfrac{1+2n}{5+3n} = \dfrac{2}{3}.$
   (iii) $\lim_{n\to\infty} \dfrac{\sin n}{n+1} = 0.$

4.2. Give the definition of divergence of a sequence without referring to converge of a sequence. Use your definition to show that the sequence $s_n = (-1)^n + \frac{1}{n}$ diverges.

4.3. Give a definition of $\lim_{n\to\infty} s_n = -\infty$. Use your definition to verify that $\lim \log_a n = -\infty$ for $0 < a < 1$.

4.4. Let the sequence $\{s_n\}$ be defined inductively as $s_1 = 1$, and $s_{n+1} = {s_n}^2 - 1$ for $n > 1$. Compute L using the ideas of Example 4.10, and then show that this L cannot be the limit of the sequence s_n.

4.5. Use the Squeeze Theorem to find $\lim\limits_{n\to\infty} \dfrac{\sin n + \cos n}{\sqrt{n}}$.

4.6. Prove that if a sequence $\{s_n\}$ converges, then the set $S = \{s_1, s_2, \dots\}$ is bounded.

4.7. Let $\{s_n\}$ be defined as $s_1 = 0.3$, $s_2 = 0.33$, $s_3 = 0.333$, ... Prove that $\{s_n\}$ converges.

4.8. Let $\{f_n\}$ be the Fibonacci sequence as defined in Example 4.2. Consider a sequence

$$s_1 = 1, \quad s_n = \frac{f_{n+1}}{f_n} \text{ for } n > 1.$$

Assume that s_n converges. Find its limit.

4.9. Show that the sequence $\{x_n\}$ defined by $x_1 = 3$, $x_{n+1} = \frac{1}{4-x_n}$ for $n > 1$, converges. Then find the limit.

4.10. Following the discussion in Example 4.15, consider the inductive sequences of the arithmetic means $\{a_n\}$ and the geometric means $\{b_n\}$ starting with two distinct positive numbers a and b with $a > b$. Show that both sequences converge to the same limit. (This limit is called the *arithmetic-geometric mean* of a and b. Surprisingly, it is a difficult problem to find exact formula for its value, in terms of a and b, in fact, this formula involves the so-called *elliptic* integrals.)