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CALC 1501 LECTURE NOTES

RASUL SHAFIKOV

5. Series

5.1. Basic Definitions. Given a sequence of real numbers

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

a formal expression

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots = \sum_{n=1}^{\infty} a_n \tag{1}$$

is called an *infinite series*, or just a *series*. We can add finitely many terms of the series to obtain

$$\begin{aligned} A_1 &= a_1, \\ A_2 &= a_1 + a_2, \\ A_3 &= a_1 + a_2 + a_3, \\ &\ldots \\ A_n &= a_1 + a_2 + a_3 + \cdots + a_n, \\ &\ldots \end{aligned}$$

The numbers $A_1, A_2, \ldots, A_n, \ldots$ are called the *partial sums* of the series (1). They naturally form a sequence $\{A_n\}$ of partial sums. If $A = \lim_{n\to\infty} A_n$ and A is a finite number, then the series $\sum a_n$ is called *convergent*, A is called its *sum*, and we write

$$A = \sum_{n=1}^{\infty} a_n.$$

If the sequence $\{A_n\}$ is divergent (i.e., A is infinite or does not exist), then the series (1) is also called *divergent*.

Example 5.1. Perhaps the simplest example of an infinite series is the so-called *geometric series*

$$a + aq + aq^2 + \cdots + aq^n + \cdots = \sum_{n=0}^{\infty} aq^{n-1}, \quad a \neq 0.$$

Its partial sum for $q \neq 1$ equals

$$A_n = \frac{a - aq^n}{1 - q}. \tag{2}$$

Indeed, by direct computation we obtain

$$A_n - qA_n = (a + aq + \cdots + aq^{n-1}) - q(a + aq + \cdots + aq^{n-1}) = a - aq^n,$$

from which equation (2) immediately follows.

By taking the limit in (2), we see that if $|q| < 1$ then the geometric series converges with the sum equal to $\frac{a}{1-q}$. If $|q| \geq 1$, then the series diverges. In particular, if $q = 1$, then $\lim A_n$ is either ∞ or $-\infty$, depending on the sign of a, and if $q = -1$, then the series takes the form

$$a - a + a - a + \dots,$$

and the value of partial sums alternates between a and 0. To summarize, the geometric series converges if $|q| < 1$, and

$$a + aq + aq^2 + \cdots + aq^n + \cdots = \sum_{n=0}^{\infty} aq^{n-1} = \frac{a}{1-q}.$$

◇

Example 5.2. Consider the series

$$\sum_{n=1}^{\infty} n\left(\frac{1}{2}\right)^n = \frac{1}{2} + \frac{1}{2} + \frac{3}{8} + \frac{1}{4} + \cdots .$$

This series resembles the geometric series, and we can try to find its sum using a similar technique. Let S_n be a partial sum of the first n terms. Then

$$S_n - \frac{1}{2}S_n = \left(\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \cdots + \frac{n}{2^n}\right) - \left(\frac{1}{2^2} + \frac{2}{2^3} + \frac{3}{2^4} + \cdots + \frac{n}{2^{n+1}}\right)$$

$$= \frac{1}{2} + \frac{2-1}{2^2} + \frac{3-2}{2^3} + \cdots + \frac{n-(n-1)}{2^n} - \frac{n}{2^{n+1}} = \left(\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n}\right) - \frac{n}{2^{n+1}}$$

The term in parentheses on the right-hand side of the above identity is the geometric series with $a = 1/2$ and $q = 1/2$, its partial sum was computed in the previous example:

$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n} = \frac{1/2 - 1/2(1/2)^n}{1 - 1/2} = 1 - (1/2)^n.$$

From this we conclude that

$$S_n = \frac{1 - (1/2)^n - n/2^{n+1}}{1/2} = 2 - \frac{1}{2^{n-1}} - \frac{n}{2^n} \tag{3}$$

It follows that S_n converges to 2 as $n \to \infty$. Thus,

$$\sum_{n=1}^{\infty} n\left(\frac{1}{2}\right)^n = 2.$$

◇

Example 5.3. Determine the convergence of the series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}.$$

We estimate its partial sum:

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} > n \cdot \frac{1}{\sqrt{n}} = \sqrt{n}.$$

We see that the partial sums grow indefinitely as n goes to infinity. Thus this series diverges. ◇

Let $\sum_{n=1}^{\infty} a_n$ be a series, and $A_m = \sum_{n=1}^{m} a_n$ be the partial sum. The quantity

$$R_m = \sum_{n=1}^{\infty} a_n - A_m = \sum_{n=m+1}^{\infty} a_n \tag{4}$$

is called the *remainder* of the series. We first observe that the series $\sum a_n$ converges if and only if any remainder R_m converges (as a series). Therefore, we may remove any finite (possibly very large!) number of elements from the series without affecting its convergence (or divergence). Further, if the series $\sum a_n$ converges, then by taking limit in (4) as $m \to \infty$ we see that $R_m \to 0$. The next theorem gives a simple test to verify divergence of certain series.

Theorem 5.1. *If the series* $\sum_{n=1}^{\infty} a_n$ *converges, then* $\lim_{n\to\infty} a_n = 0$.

Proof. Let $A_n = a_1 + \cdots + a_n$. Then, since the series $\sum a_n$ converges, $\lim A_n$ exists as $n \to \infty$. Hence, $a_n = A_n - A_{n-1}$, and so $\lim_{n\to\infty} a_n = \lim_{n\to\infty}(A_n - A_{n-1}) = 0$. □

The *contrapositive* formulation of this theorem is sometimes called the *Test for Divergence*: if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum a_n$ diverges. For example, the series

$$\sum_{n=1}^{\infty} \frac{1}{1+s^n}$$

diverges for $0 < s \leq 1$ because $1/(1+s^n)$ does not converge to zero as $n \to \infty$. It is, however, wrong in general to conclude from the convergence of $\{a_n\}$ to zero that the series $\sum a_n$ converges. For instance, in Example 5.3 the series diverges, yet $\lim a_n = 0$.

Suppose now that the series $\sum a_n$ consists of positive terms. Then partial sums $\{A_n\}$ form an increasing sequence. If this sequence is bounded, then by the Monotone Convergence Theorem, it follows that the sequence of partial sums (and therefore the series) converges. On the other hand, if the sequence of partial sums is unbounded, then the series diverges. We illustrate this in the next example.

Consider the so-called *harmonic series*[1] given by

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n}. \tag{5}$$

Indeed, starting from the third term we can divide the series into groups consisting of $2, 4, 8, \ldots, 2^k, \ldots$ terms in each group:

$$\underbrace{\frac{1}{3} + \frac{1}{4}}_{2} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}_{4} + \underbrace{\frac{1}{9} + \cdots + \frac{1}{16}}_{8} + \cdots.$$

[1]The reason for the name is that every term of the series is the *harmonic mean* of the two neighbouring terms. Recall that the harmonic mean of two numbers a and b equals $\frac{2ab}{a+b}$. The harmonic mean is an important notion in geometry and physics.

Each group adds up to a number bigger than 1/2. Therefore, if we denote by H_n the partial sum of the first n terms of the series, we see that

$$\begin{aligned} &H_4 > 1/2 + 1/2 = 1 \\ &H_8 > H_4 + 1/2 > 1 + 1/2 = 3/2 \\ &H_{16} > H_8 + 1/2 > 3/2 + 1/2 = 2 \\ &\dots \\ &H_{2^k} > k \cdot 1/2 \\ &\dots \end{aligned} \tag{6}$$

Thus the sequence of partial sums is unbounded, and the harmonic series diverges. We note that as n grows, the value of the partial sum of n terms grows rather slowly. For example, Euler calculated that $H_{1000} \approx 7.48$ and $H_{1000000} = 14.39$.

Let us consider a more general series of the form

$$1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \dots = \sum_{n=1}^{\infty} \frac{1}{n^s}, \tag{7}$$

where s is some positive real number. If $s = 1$, then (7) becomes the harmonic series. If $s < 1$, then the terms in (7) are bigger than the corresponding terms in (5), and so are the partial sums, hence, the series also diverges.

Now consider the case $s > 1$. We write $s = 1 + t$, where t is some positive number. We have

$$\frac{1}{(n+1)^s} + \frac{1}{(n+2)^s} + \dots + \frac{1}{(2n)^s} < n \cdot \frac{1}{n^s} = \frac{1}{n^t}. \tag{8}$$

Splitting the series into groups, analogously to what we did for the harmonic series we have

$$\underbrace{\frac{1}{3^s} + \frac{1}{4^s}}_{2} + \underbrace{\frac{1}{5^s} + \frac{1}{6^s} + \frac{1}{7^s} + \frac{1}{8^s}}_{4} + \underbrace{\frac{1}{9^s} + \dots + \frac{1}{16^s}}_{8} + \dots .$$

From (8) it follows that each group above is less than the corresponding term of the geometric series

$$\left\{ \frac{1}{2^t}, \frac{1}{4^t}, \frac{1}{8^t}, \dots \right\} = \left\{ \frac{1}{2^t}, \frac{1}{(2^t)^2}, \frac{1}{(2^t)^3}, \ \dots \right\}$$

Since this geometric series $\{(\frac{1}{2^t})^n\}$ converges, we conclude that the sequence of partial sums of the series in (7) is bounded above, and therefore converges by the Monotone Convergence Theorem. Hence, the series (7) also converges. Another proof of convergence of the series for $s > 1$ will be given later, when we discuss the Integral Test for Convergence. The sums of this series are the values of a famous function $\zeta(s)$, called the Riemann ζ-function. It plays fundamental role in Number theory.

Example 5.4. Consider

$$\sum_{n=2}^{\infty} \frac{1}{n^2 - n}.$$

First observe that $\frac{1}{n^2-n} = \frac{1}{n-1} - \frac{1}{n}$. Therefore, the partial sum A_n of this series equals

$$\begin{aligned} A_n &= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) \\ &= 1 + \left(-\frac{1}{2} + \frac{1}{2}\right) + \cdots + \left(-\frac{1}{n-1} + \frac{1}{n-1}\right) - \frac{1}{n} = 1 - \frac{1}{n}. \end{aligned} \tag{9}$$

Thus $A_n \to 1$, and the series converges to 1. Series of this type are called *telescoping series.* $\diamond$

5.2. **Comparison Theorems.** Convergence or divergence of series can be often determined by comparing a given series to another series, which is known to converge or diverge. In the next theorems we assume that $\sum_n^\infty a_n$, and $\sum_n^\infty b_n$ are series with positive terms

Theorem 5.2 (Comparison Test)**.** *Suppose that there exists a number $N > 0$ such that the inequality $a_n \le b_n$ holds for all $n > N$. Then convergence of $\sum b_n$ implies convergence of $\sum a_n$. Equivalently, divergence of $\sum a_n$ implies that of $\sum b_n$.*

Proof. We may drop any finite number of terms of the series without affecting its convergence. Therefore, without loss of generality we may assume that that $a_n \le b_n$ for all $n = 1, 2, \ldots$. Denote by A_n, and B_n the partial sums of $\sum a_n$ and $\sum b_n$ respectively. Then $A_n \le B_n$. Suppose that $\sum b_n$ converges. Then the sequence of partial sums $\{B_n\}$ is bounded above: $B_n \le L$, for some $L > 0$. Therefore $A_n \le B_n \le L$, and by the Monotone Convergence Theorem, the sequence $\{A_n\}$ also converges. This proves the theorem. $\square$

Theorem 5.3 (Limit Comparison Test)**.** *Suppose there exists a limit*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = K \quad (0 \le K \le \infty).$$

Then:

(i) if the series $\sum b_n$ converges and $K < \infty$, then $\sum a_n$ converges.
(ii) if $\sum b_n$ diverges and $K > 0$, then $\sum a_n$ also diverges.

Proof. Suppose $\sum b_n$ converges with $K < \infty$. Given any $\epsilon > 0$ by definition of the limit, for sufficiently large n we have

$$\frac{a_n}{b_n} < K + \epsilon \implies a_n < (K + \epsilon) \cdot b_n$$

Since the series $\sum c_n = \sum (K + \epsilon) b_n$ obtained by multiplying the series $\sum b_n$ by a constant $(K + \epsilon)$ converges, we may apply the Comparison Test to $\sum a_n$ and $\sum c_n$ to conclude that the series $\sum a_n$ also converges.

The proof of the second statement is Exercise 6.5. $\square$

Theorem 5.4. *Suppose there exists $N > 0$ such that for $n > N$ we have*

$$\frac{a_{n+1}}{a_n} \le \frac{b_{n+1}}{b_n}. \tag{10}$$

Then convergence of $\sum b_n$ implies convergence of $\sum a_n$ and divergence of $\sum a_n$ implies that of $\sum b_n$.

The proof of this theorem is Exercise 6.7.

Example 5.5. Test for convergence the following series.

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$,

(ii) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\sqrt[n]{n}}$,

(iii) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n} - \ln\frac{n+1}{n}\right)$.

Solutions:

(i) $\dfrac{n!}{(2n)!} = \dfrac{1^2 2^2 3^2 \dots n^2}{1\cdot 2\cdot \ldots \cdot n\cdot (n+1)\cdot \ldots \cdot (2n)} = \dfrac{1}{n+1}\cdot\dfrac{2}{n+2}\cdot \ldots \cdot \dfrac{n}{2n} < \dfrac{1}{2^n}$. Since $\sum \frac{1}{2^n}$ converges, it follows by the Comparison Test (Theorem 5.2) that the series $\sum \frac{n!}{(2n)!}$ also converges.

(ii) We use the Limit Comparison Theorem: Since

$$\frac{1}{n\sqrt[n]{n}} \div \frac{1}{n} = \frac{1}{\sqrt[n]{n}} \to 1,$$

and the harmonic series $\sum \frac{1}{n}$ diverges, we conclude that the series $\sum \frac{1}{n\sqrt[n]{n}}$ also diverges.

(iii) We use the inequality $\ln(1+x) \le x$, which holds for $-1 < x$ (See Lecture 1, Example. 1.12). First observer that

$$\ln\left(1+\frac{1}{n}\right) < \frac{1}{n} \quad\Longrightarrow\quad 0 < \frac{1}{n} - \ln\left(\frac{n+1}{n}\right).$$

Furthermore,

$$-\ln\frac{n+1}{n} = \ln\frac{n}{n+1} = \ln\left(1-\frac{1}{n+1}\right) < -\frac{1}{n+1}.$$

Therefore,

$$0 < \frac{1}{n} - \ln\frac{n+1}{n} < \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)} < \frac{1}{n^2}$$

Thus, the series converges by the Comparison Test and convergence of $\sum \frac{1}{n^2}$. ⋄

Exercises

5.1. Use the technique of Example 5.2 to find the values of q for which the series

$$\sum_{n=1}^{\infty} nq^n$$

converges.

5.2. Prove that if the series $\sum a_n$ converges then its remainder R_m as defined in (4) converges to zero.

5.3. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+s^n}$ converges or diverges for $s > 1$.

5.4. Find the some of the series if it is converging: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+3)}$.

5.5. Prove part (ii) of Theorem 5.3.

5.6. Test for convergence the following series:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n+2)}}$,

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$,

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(\ln n)^p}, \quad p > 0,$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(\ln n)^{\ln n}}.$

5.7. Prove Theorem 5.4. Hint: multiply equations (10) term by term, and use Theorem (5.2).