

CALC 1501 LECTURE NOTES

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6. POWER SERIES AND TAYLOR SERIES

6.1. Power series. A power series *centred at a point* a is defined to be an expression of the form

$$(1) \quad \sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \dots,$$

where c_j are some numbers. In particular, a power series centred at the origin (i.e., when $a = 0$) has the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots$$

A power series centred at 0 can be thought of as an "infinite polynomial". The values of x for which the series (1) converges form *the domain of convergence* of the power series. It is clear from (1) that it always contains its centre a , and so on its domain of convergence the power series defines a function of x .

Example 6.1. Consider the power series centred at the origin given by

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots$$

This is a geometric series with $a = 1$, $q = x$. The series converges for $|x| < 1$, and diverges for all other values of x . In fact, the function that this series defines on $|x| < 1$ is

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$$

We will show that the domain of convergence of a power series is always an interval (finite or infinite), or one point - its centre. For that we first prove the following

Lemma 6.1. *If the power series*

$$(2) \quad \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots$$

converges for some $x = b$, then it converges for all $|x| < |b|$.

Proof. Since the power series (2) converges for $x = b$, the sequence $\{c_n b^n\}$ is bounded, i.e., there exists $M > 0$ such that $|c_n b^n| < M$ for all n . Therefore, for any x satisfying $|x| < |b|$, we have

$$|c_n x^n| = \left| \frac{c_n x^n b^n}{b^n} \right| < M \left| \frac{x^n}{b^n} \right| = M \left| \frac{x}{b} \right|^n.$$

Since $|x/b| < 1$, the geometric series $\sum_{n=0}^{\infty} M \left| \frac{x}{b} \right|^n$ converges, and by the Comparison Test, so does

the series $\sum_{n=0}^{\infty} c_n x^n$. □

Example 6.2. Consider $\sum_{n=0}^{\infty} n! x^n$. By the Ratio Test

$$\lim_{n \rightarrow \infty} \frac{(n+1)! x^{n+1}}{n! x^n} = \lim_{n \rightarrow \infty} (n+1)x = \infty$$

for all values of x . Therefore, the series converges only for $x = 0$. \diamond

If the domain of convergence is unbounded, then it must equal all of \mathbb{R} . Indeed, for all points b in the domain of convergence, by Lemma 6.1 the series converges for all x which are less than $|b|$ in absolute value. By taking bigger and bigger b , we see that the series converges for all $x \in \mathbb{R}$.

On the other hand, suppose that the series in (2) has a point different from the origin where it converges, and that the domain of convergence is bounded. Consider the least upper bound of the set of values of x for which the series converges. Recall that the least upper bound exists by the Axiom of Completeness (see Lecture 4). Denote it by R . We claim that if $|x| < R$, then the series converges. Indeed, suppose on the contrary that the series diverges for this x . It follows from Lemma 6.1 that for $|x'| > |x|$ the series also diverges. So $|x|$ is an upper bound for the domain of convergence of the series. But this contradicts the fact that R is the least upper bound. This proves the claim. Similarly, if $|x| > R$, then the series diverges. Combining everything together yields the following

Theorem 6.2. For the series

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots$$

one of the following mutually exclusive possibilities holds:

- (i) The series converges at the origin only.
- (ii) The series converges on the whole real line \mathbb{R} .
- (iii) There exists $R > 0$ such that the series converges for $|x| < R$ and diverges for $|x| > R$.

The number R in (iii) is called *the radius of convergence* of the series. We will use the convention that $R = 0$ in (i), and $R = \infty$ in (ii). When possibility (iii) holds the power series may converge at the end points R and $-R$. Thus, the domain of convergence in this case is one of the following intervals: $(-R, R)$, $[-R, R)$, $(-R, R]$, or $[-R, R]$. Because of that, we will also call the domain of convergence *the interval of convergence* of the power series.

Corollary 6.3. The power series

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \dots$$

either converges at one point a , or on the whole of \mathbb{R} , or on one of the bounded interval $(a-R, a+R)$, $[a-R, a+R)$, $(-aR, a+R]$, or $[a-R, a+R]$, where $R > 0$.

Example 6.3. Find the interval of convergence of the following power series:

$$(a) \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$(b) \sum_{n=1}^{\infty} \frac{(x-5)^n}{n 2^n}$$

Solution: (a) We apply the Ratio Test:

$$\lim_{n \rightarrow \infty} \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} = \lim_{n \rightarrow \infty} \frac{x}{n+1} = 0$$

for all x . Thus, the interval of convergence of this power series is \mathbb{R} .

(b) Again by the Ratio test,

$$\lim_{n \rightarrow \infty} \frac{|x-5|^{n+1}}{(n+1)2^{n+1}} \cdot \frac{n2^n}{|x-5|^n} = \frac{|x-5|}{2}.$$

For the convergence of the series we need to have $\frac{|x-5|}{2} < 1$, or

$$|x-5| < 2.$$

Thus the radius of convergence equals 2. To determine the interval of convergence we need to investigate the end points. When $x = -3$, the series becomes

$$\sum_{n=1}^{\infty} \frac{(3-5)^n}{n 2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n},$$

which converges by the Alternating Series Test. For the other end point $x = 7$ we have

$$\sum_{n=1}^{\infty} \frac{(7-5)^n}{n 2^n} = \sum_{n=1}^{\infty} \frac{1}{n},$$

which is a harmonic series, and so diverges. Thus the interval of convergence of the series is $[3, 7)$. \diamond

Sometimes it is convenient to find the radius of convergence R of the power series $\sum c_n(x-a)^n$ using the so-called *Cauchy-Hadamard formula*:

$$(3) \quad \frac{1}{R} = \lim_{n \rightarrow \infty} \sqrt[n]{|c_n|},$$

provided that the limit exists. (If the limit does not exist, then one can show that $1/R$ equals the *largest* limit of $\sqrt[n]{|c_n|}$, that is the limit as n approaches infinity of the supremum of the elements of the sequence after the n -th position.)

Example 6.4. To find the interval of convergence of the series

$$\sum_{n=1}^{\infty} 3^n x^n$$

we use formula (3):

$$\lim_{n \rightarrow \infty} \sqrt[n]{|3^n|} = 3,$$

and so $R = 1/3$. A simple calculation shows that the series diverges at both end points, $x = 1/3$ and $x = -1/3$. Thus the interval of convergence is $(-1/3, 1/3)$. \diamond

One can show that on the (open) interval of convergence, the power series (1) is continuous and differentiable. The proof of these facts, however, goes beyond our capabilities at this point. Furthermore, the convergent power series can be differentiated and integrated term-by-term. This will produce the functions which are respectively the derivative and the anti-derivative of the function defined by the original series. More precisely, the following holds.

Theorem 6.4. *Suppose the power series $\sum_n c_n(x-a)^n$ has the radius of convergence $R > 0$, and on the interval $(a-R, a+R)$ it defines the function $f(x)$:*

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n.$$

Then

$$(4) \quad f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1},$$

and

$$(5) \quad \int f(x)dx = C + c_0(x-a) + c_1 \frac{(x-a)^2}{2} + c_2 \frac{(x-a)^3}{3} + \cdots = C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}.$$

Furthermore, the radius of convergence is the same for all three power series above.

The theorem can be proved by applying the Root Test to the series in (4) and (5), and using the fact that $\sqrt[n]{n} \rightarrow 1$ as $n \rightarrow \infty$. It should be noted that in the theorem above, while the radius of convergence is the same, the *interval of convergence* can be different for $f(x)$, $f'(x)$ and $\int f(x)dx$.

Example 6.5. Find the interval of convergence of the power series

$$\sum_{n=1}^{\infty} nx^n,$$

and compute the function that it represents.

Solution: we consider the series $\sum_{n=1}^{\infty} x^n$. For every fixed x with $|x| < 1$, this is a geometric series converging to $\frac{x}{1-x}$:

$$\sum_{n=1}^{\infty} x^n = x + x^2 + x^3 + \cdots = \frac{x}{1-x}$$

Differentiating this series term-by-term gives

$$\sum_{n=1}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2}.$$

Therefore, we conclude that

$$\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}.$$

By Theorem 6.4 this series has the radius of convergence 1. By inspection, the series diverges at both end points $x = 1$ and $x = -1$. Thus the interval of convergence of the series is $(-1, 1)$. \diamond

Example 6.6. Find the power series representation for the functions $f(x) = \frac{1}{2+x}$ centred at (a) the origin, (b) $a = 1$.

(a) We have

$$\frac{1}{2+x} = \frac{1}{2} \cdot \frac{1}{1 - (-x/2)} = \frac{1}{2} \sum_{n=0}^{\infty} \left(-\frac{x}{2}\right)^n = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{2^{n+1}} = \frac{1}{2} - \frac{x}{4} + \frac{x^2}{8} - \frac{x^3}{16} + \dots$$

Using the ratio test one can see that the interval of convergence of this power series is $(-2, 2)$.

(b) The power series representation centred at $a = 1$ will have the powers of $(x - 1)$. Therefore, we should try to use the geometric series that will have the powers of $(x - 1)$. We have

$$\frac{1}{2+x} = \frac{1}{3+(x-1)} = \frac{1}{3} \cdot \frac{1}{1 - (-\frac{x-1}{3})} = \frac{1}{3} \sum_{n=0}^{\infty} \left(-\frac{x-1}{3}\right)^n = \sum_{n=0}^{\infty} (-1)^n \frac{(x-1)^n}{3^{n+1}}.$$

Again, using the Ratio Test, we see that the radius of convergence is 3, and the interval of convergence is $(-2, 4)$. Note that although the power series in (a) and (b) are different, and have different intervals of convergence, the point $x = -2$ is the end point for both of them. This is because the function $f(x)$ is not defined at this point. \diamond

Example 6.7. To find the power series representation of the function $\ln(1+x^2)$ centred at the origin, first consider the function $\ln(1+x)$. Its derivative equals $\frac{1}{1+x}$ with

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = 1 - x + x^2 - x^3 + x^4 - \dots = \sum_{n=0}^{\infty} (-1)^n x^n.$$

Integrating the above expression term-by-term we obtain

$$\ln(1+x) = \int \frac{dx}{1+x} = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + C = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} + C.$$

To determine the value of C , we set $x = 0$ which yields $0 = \ln(1+0) = C$. Therefore,

$$(6) \quad \ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}.$$

Finally, by replacing x with x^2 in the above formula, we obtain:

$$\ln(1+x^2) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x^2)^n}{n} = x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \dots$$

\diamond

Example 6.8. Prove that

$$\ln 2 = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

Solution: Substitute $x = 1$ into formula (6) to get the result.

Note: This seemingly simple solution, however, requires a justification: why is the power series continuous (from one side) at the end point of the interval of convergence, provided that the series converges at that point? This is the content of the so called Abel's theorem, the proof of which can be found in advanced calculus textbooks. \diamond

6.2. Taylor Series. Let $f(x)$ be a function that has derivatives of all orders on the interval $(a - R, a + R)$ for some $a \in \mathbb{R}$, and $R > 0$. Suppose that $f(x)$ can be represented on $(a - R, a + R)$ by a convergent power series

$$(7) \quad \sum_{n=0}^{\infty} c_n (x - a)^n.$$

This means that for any $x \in (a - R, a + R)$, the series (7) converges to $f(x)$. Then by direct differentiation of the power series (7), we see that $f^{(n)}(a) = n!c_n$, for all $n > 0$ (here $f^{(n)}$ denotes the derivative of $f(x)$ of order n). From this we conclude that

$$c_n = \frac{f^{(n)}(a)}{n!},$$

and thus the series in (7) becomes

$$(8) \quad \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!} (x - a)^2 + \dots$$

This is called the *Taylor series centred at $x = a$ associated with $f(x)$* . If $a = 0$, then (8) becomes

$$(9) \quad \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0)x + \frac{f''(0)}{2!} x^2 + \dots,$$

which is called the *Maclaurin series* associated with $f(x)$.

Example 6.9. Let $P(x)$ be a polynomial of degree N ,

$$P(x) = c_0 + c_1x + c_2x^2 + \dots + c_Nx^N.$$

By inspection, $c_n = \frac{P^{(n)}(0)}{n!}$ for $n = 1, \dots, N$, and $c_n = 0$ for $n > N$. Thus, the Maclaurin series associated with $P(x)$ is exactly $P(x)$. \diamond

In general, however, one cannot immediately conclude that the Taylor or Maclaurin series associated with a function $f(x)$ converges to $f(x)$. In fact, it is not even clear whether the Taylor series of a given function converges at all. (Note that when we derived (8) we assumed to begin with that $f(x)$ has a power series representation.) Define the *Taylor polynomial* to be

$$(10) \quad T_N(x) = \sum_{n=0}^N \frac{f^{(n)}(a)}{n!} (x - a)^n = f(a) + f'(a)(x - a) + \dots + \frac{f^{(N)}(a)}{N!} (x - a)^N,$$

i.e., $T(x)$ is simply the order N partial sum of the Taylor series (8). Thus, by the definition of convergence, in order to show convergence of the Taylor series to $f(x)$ we need to show that

$$(11) \quad \lim_{N \rightarrow \infty} T_N(x) = f(x)$$

for all x on some interval. If we define the *remainder* of the Taylor series to be

$$(12) \quad R_N(x) = f(x) - T_N(x)$$

then proving (11) is equivalent to showing

$$R_N(x) \rightarrow 0, \quad \text{as } N \rightarrow \infty.$$

The following theorem provides a useful tool for proving convergence of Taylor series. For simplicity, we consider the case when $a = 0$. Then $T_N(x) = f(0) + f'(0)x + \dots + \frac{f^{(N)}(0)}{N!}x^N$, and

$$(13) \quad R_N(x) = f(x) - f(0) + f'(0)x + \dots + \frac{f^{(N)}(0)}{N!}x^N$$

Theorem 6.5 (Lagrange's Remainder Theorem). *Let f be infinitely differentiable on $(-R, R)$. Then there exists a number c satisfying $|c| < |x|$ such that*

$$(14) \quad R_N(x) = \frac{f^{(N+1)}(c)}{(N+1)!}x^{N+1}.$$

Example 6.10. Let $f(x) = e^x$. Then $f^{(n)}(0) = e^0 = 1$ for all n . Therefore, $c_n = \frac{1}{n!}$, and we have

$$e^x \sim \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

The remainder of order N of this Maclaurin series is

$$R_N = \frac{x^{N+1}}{(N+1)!} + \frac{x^{N+2}}{(N+2)!} + \dots$$

According to Lagrange's Remainder Theorem, there is a number c , $|c| < |x|$, such that

$$R_N(x) = \frac{f^{(N+1)}(c)}{(N+1)!}x^{N+1} = \frac{e^c}{(N+1)!}x^{N+1}.$$

For any fixed x , $R_N(x) \rightarrow 0$, since for any x , $\frac{x^n}{n!} \rightarrow 0$ as $n \rightarrow \infty$. Thus

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

for all $x \in \mathbb{R}$. \diamond

Example 6.11. Let

$$(15) \quad g(x) = \begin{cases} e^{-1/x^2}, & \text{if } x > 0 \\ 0, & \text{if } x \leq 0 \end{cases}$$

Since e^{-1/x^2} approaches 0 as $x \rightarrow 0$, the function $g(x)$ is continuous at 0. In fact, using L'Hôpital's Rule one can show that $g(x)$ has continuous derivatives of any order at $x = 0$, and $g^{(n)}(0) = 0$ for any $n > 0$. The Maclaurin series associated to $g(x)$ is, therefore, identically zero. It follows that the Maclaurin series associated with $g(x)$ does not converge to $g(x)$ for $x > 0$. \diamond

Definition 6.6. *An infinitely differentiable function $f(x)$ is called real-analytic in a neighbourhood of a point $x = a$, if for some positive R the Taylor series (8) associated with $f(x)$ converges to $f(x)$ on $(a - R, a + R)$.*

Thus, e^x is a real-analytic function, while the function $g(x)$ in Example 6.11 is not real analytic near $x = 0$.

Proof of Lagrange's Remainder Theorem. . First note the following version of the Mean Value Theorem: If $g(x)$ and $h(x)$ are continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b) and $h'(x) \neq 0$, then there exists a point $c \in (a, b)$ such that

$$(16) \quad \frac{g(b) - g(a)}{h(b) - h(a)} = \frac{g'(c)}{h'(c)}$$

This can be proved by applying the Mean Value Theorem to the function

$$\phi(x) = (g(b) - g(a))h(x) - (h(b) - h(a))g(x).$$

Note that the n -th order derivative of $R_N(x)$ at $x = 0$ vanishes for $n = 0, 1, 2, \dots, N$ (see Exercise 6.2). Therefore, if we apply (16) to functions $g(x) = R_N(x)$ and $h(x) = x^{N+1}$, then (assume $x > 0$ for simplicity) there exists a point $c_1 \in (0, x)$ such that

$$\frac{R_N(x)}{x^{N+1}} = \frac{R'_N(c_1)}{(N+1)c_1^N}.$$

We now repeat the process and apply (16) to functions $g(x) = R'_N(x)$ and $h(x) = x^N$ on the interval $(0, c_1)$: there is $c_2 \in (0, c_1)$ such that

$$\frac{R'_N(c_1)}{c_1^N} = \frac{R''_N(c_2)}{Nc_2^{N-1}}.$$

Continue the process inductively N times. In the end we get

$$R_N(x) = \frac{x^{N+1}}{(N+1)!} \frac{R_N^{(N+1)}(c_{N+1})}{c_{N+1}^{N-N}},$$

where $c_{N+1} \in (0, c_N) \subset \dots \subset (0, x)$. Now set $c = c_{N+1}$, then $c^{N-N} = 1$, and we can write

$$R_N(x) = \frac{R_N^{(N+1)}(c)}{(N+1)!} x^{N+1} = \frac{f^{(N+1)}(c)}{(N+1)!} x^{N+1},$$

where the last equality follows from the fact that $R_N^{(N+1)}(x) = (f(x) - T_N(x))^{(N+1)} = f^{(N+1)}(x)$, because $T_N^{(N+1)} \equiv 0$. This proves the theorem. \square

Example 6.12. Let $f(x) = (1+x)^{1/2}$. Then

$$f^{(n)}(0) = \frac{1}{2} \left(\frac{1}{2} - 1\right) \left(\frac{1}{2} - 2\right) \dots \left(\frac{1}{2} - n + 1\right).$$

Therefore,

$$c_n = \binom{1/2}{n} = \frac{\frac{1}{2} \left(\frac{1}{2} - 1\right) \left(\frac{1}{2} - 2\right) \dots \left(\frac{1}{2} - n + 1\right)}{n!},$$

and hence

$$(1+x)^{1/2} \sim \sum_{n=0}^{\infty} \binom{1/2}{n} x^n$$

is the associated Maclaurin series. This is called the *binomial series*. Let us try use Lagrange's Remainder Theorem again to determine convergence of the series above. We have

$$R_N(x) = \frac{\frac{1}{2} \left(\frac{1}{2} - 1\right) \left(\frac{1}{2} - 2\right) \dots \left(\frac{1}{2} - N\right) (1+c)^{1/2-N}}{(N+1)!} x^{N+1}$$

for some c , $|c| < |x|$. If $|x| < 1$, then clearly $x^{N+1} \rightarrow 0$ as $N \rightarrow \infty$. Also, $\lim_{N \rightarrow \infty} \binom{1/2}{N} = 0$ (see Exercise 6.6). If $c > 0$, then we also have $(1+c)^{1/2-N} \rightarrow 0$ as $N \rightarrow \infty$. However, if $c < 0$, then $(1+c)^{1/2-N}$ does not go to zero, and we cannot be sure that $R_N(x)$ goes to zero.

In general, the binomial series converges for $x \in (-1, 1)$, and we have

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n, \quad k \in \mathbb{R}, \quad \text{and } |x| < 1,$$

where the *binomial coefficients* are defined by

$$\binom{k}{n} = \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}.$$

◇

Exercises

- 6.1. Formulate Lemma 6.1 for the power series given by equation (1).
- 6.2. Prove that in the proof of Lagrange's theorem the remainder $R_N(x)$ given by equation (13) satisfies $R_N^{(n)}(0) = 0$ for all $n = 0, 1, \dots, N$.
- 6.3. Formulate Theorem 6.2 for the power series given by equation (1).
- 6.4. Show that the function $g(x)$ in Example 6.11 satisfies $g'(0) = 0$.
- 6.5. Use Lagrange's Remainder Theorem to prove that the Maclaurin series of $\cos x$ converges to $\cos x$ for all x .
- 6.6. Show that for any m ,

$$\lim_{n \rightarrow \infty} \binom{m}{n} = 0.$$