Solutions to Practice Midterm 2

Problem 1. Evaluate $\int_0^1 x \arctan\left(x^2\right)\, dx$.

Solution. We integrate by parts using the following:

$$u = \arctan\left(x^2\right), \qquad dv = x dx, \qquad du = \frac{2x}{1+x^4} dx, \qquad v = \frac{x^2}{2}.$$

We then have

$$\int_0^1 x \arctan\left(x^2\right)\, dx = \frac{x^2}{2} \arctan\left(x^2\right)\Big|_0^1 - \int_0^1 \frac{x^3}{1+x^4}\, dx$$

We then substitute $u = 1 + x^4$, $du = 4x^3 dx$.

$$\begin{aligned}
\frac{x^2}{2} \arctan\left(x^2\right)\Big|_0^1 - \int_0^1 \frac{x^3}{1+x^4}\, dx &= \frac{x^2}{2} \arctan\left(x^2\right)\Big|_0^1 - \frac{1}{4} \int_{u(0)}^{u(1)} \frac{1}{u}\, dx \\
&= \frac{x^2}{2} \arctan\left(x^2\right)\Big|_0^1 - \frac{1}{4} \ln(u)\Big|_{u(0)}^{u(1)} \\
&= \frac{x^2}{2} \arctan\left(x^2\right)\Big|_0^1 - \frac{1}{4} \ln\left(1+x^4\right)\Big|_0^1 \\
&= \frac{\pi}{8} - \frac{1}{4} \ln 2.
\end{aligned}$$

Problem 2. Evaluate $\int e^x \sin x\, dx$.

Solution. We integrate by parts twice. Let

$$u = e^x, \qquad dv = \sin x\, dx, \qquad du = e^x dx, \qquad v = -\cos x.$$

We then have

$$\int e^x \sin x\, dx = -e^x \cos x + \int e^x \cos x\, dx.$$

To integrate $\int e^x \cos x\, dx$, let

$$u = e^x, \qquad dv = \cos x\, dx, \qquad du = e^x dx, \qquad v = \sin x.$$

We have

$$\int e^x \cos x\, dx = e^x \sin x - \int e^x \sin x\, dx.$$

Substituting this into the first equation,

$$\begin{aligned}\int e^x \sin x\, dx &= -e^x \cos x + \int e^x \cos x\, dx \\ &= -e^x \cos x + e^x \sin x - \int e^x \sin x\, dx.\end{aligned}$$

This rearranges to

$$\begin{aligned}2\int e^x \sin x\, dx &= -e^x \cos x + e^x \sin x + \tilde{C} \\ \int e^x \sin x\, dx &= \frac{1}{2}\left(-e^x \cos x + e^x \sin x\right) + C.\end{aligned}$$

Problem 3. Evaluate $\int \frac{x^2+x}{(x-1)^3}\, dx$.

Solution. We integrate by partial fractions. The partial fraction decomposition is

$$\frac{x^2+x}{(x-1)^3} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3}$$

$$x^2 + x = A(x-1)^2 + B(x-1) + C.$$

Solving this equation yields $A = 1$, $B = 3$, and $C = 2$. Substituting these values into the partial fraction decomposition allows us to integrate:

$$\begin{aligned}\int \frac{x^2+x}{(x-1)^3}\, dx &= \int \frac{1}{x-1}\, dx + \int \frac{3}{(x-1)^2}\, dx + \int \frac{2}{(x-1)^3}\, dx \\ &= \ln|x-1| - 3\left(\frac{1}{x-1}\right) - \frac{1}{(x-1)^2} + K.\end{aligned}$$

Problem 4. Evaluate $\int \frac{2x^2+x+5}{x(x^2-2x+5)}\, dx$.

Solution. We integrate by partial fractions. We first complete the square in the denominator, rewriting $x^2 - 2x + 5$ as $(x-1)^2 + 4$, and then rewriting the quotient in terms of $x - 1$.

$$\frac{2x^2+x+5}{x(x^2-2x+5)} = \frac{2(x-1)^2 + 5(x-1) + 8}{([x-1]+1)\left([x-1]^2+4\right)}$$

Let $y = x - 1$. The partial fraction decomposition is

$$\frac{2y^2+5y+8}{(y+1)\left(y^2+4\right)} = \frac{A}{y+1} + \frac{By+C}{y^2+4}$$

$$2y^2 + 5y + 8 = (y^2+4)A + (y+1)(By+C).$$

Solving for A, B, and C yields $A = 1$, $B = 1$, and $C = 4$. Then,

$$\begin{aligned}\int \frac{2y^2+5y+8}{(y+1)\left(y^2+4\right)}\, dy &= \int \frac{1}{y+1}\, dy + \int \frac{y+4}{y^2+4}\, dy \\ &= \int \frac{1}{y+1}\, dy + \int \frac{y}{y^2+4}\, dy + \int \frac{4}{y^2+4}\, dy \\ &= \ln|y+1| + \frac{1}{2}\ln|y^2+4| + \arctan\frac{y}{2} + K \\ &= \ln|x| + \frac{1}{2}\ln|(x-1)^2+4| + \arctan\frac{x-1}{2} + K.\end{aligned}$$

Problem 5. Prove, using the Mean Value Theorem: If $x < 0$, then $e^x > 1 + x$.

Solution. Let $f(x) = e^x - x - 1$. Then, $f'(x) = e^x - 1$. When $x < 0$, $e^x < 1$ and $f'(x) < 0$. For every $x < 0$, $f(x)$ is continuous on $[x, 0]$ and is differentiable on $(x, 0)$; $f(x)$ satisfies the conditions of the Mean Value Theorem. Therefore, there exists $c \in (x, 0)$ such that

$$f'(c) = \frac{f(0) - f(x)}{0 - x}.$$

This evaluates to

$$e^c - 1 = \frac{e^0 - (0) - 1 - (e^x - x - 1)}{-x} = \frac{-e^x + x + e^0}{-x} = \frac{-e^x + x + 1}{-x},$$

which rearranges to

$$\underbrace{\overbrace{(-x)}^{\text{positive}}\ \overbrace{(e^c - 1)}^{\text{negative}}}_{\text{negative}} = -e^x + x + 1.$$

Multipltying through by (-1),

$$e^x - x - 1 = x(e^c - 1) > 0,$$

which implies that for all $x < 0$

$$e^x > x + 1.$$

Problem 6. Suppose that f is continuous on $[1, 3]$ and differentiable on $(1, 3)$. Futher suppose that $f(1) = 7$ and $f'(x) < 1$ for all $x \in (1, 3)$. Prove, using the Mean Value Theorem, that $f(3) < 9$.

Solution. Since f is continuous on $[1, 3]$ and differentiable on $(1, 3)$, it satisfies the conditions of the mean value theorem. So, there exists some $c \in (1, 3)$ such that

$$f'(c) = \frac{f(3) - f(1)}{3 - 1}.$$

Since $f'(x) < 1$ for all $x \in (1, 3)$, $f'(c) < 1$. Rearranging shows

$$\begin{aligned} f(3) &= (3 - 1)f'(c) + f(1) \\ &= 2f'(c) + f(1) \\ &< 2(1) + 7 < 9. \end{aligned}$$

Problem 7. Recall that the Gamma function is defined as $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt$.

(a) Use this definition of the Gamma function to evaluate $\Gamma(2)$.

(b) Determine the value of the (convergent) improper integral $\int_0^\infty t^{30}e^{-t}\,dt$.

Solution. (a) We integrate by parts.

$$\Gamma(2) = \int_0^\infty t^{2-1}e^{-t}\,dt = \int_0^\infty te^{-t}\,dt.$$

Let

$$u = t, \qquad dv = e^{-t}\,dt, \qquad du = dt, \qquad v = -e^{-t}.$$

We then have

$$\begin{aligned}\int_0^\infty te^{-t}\,dt &= -te^{-t}\Big|_0^\infty + \int_0^\infty e^{-t}\,dt \\ &= -te^{-t}\Big|_0^\infty - e^{-t}\Big|_0^\infty \\ &= -\lim_{s\to\infty} se^{-s} + 1.\end{aligned}$$

To evaluate $\lim_{s\to\infty} se^{-s}$ we use L'Hopitals Rule.

$$\begin{aligned}\Gamma(2) &= -\lim_{s\to\infty} se^{-s} + 1 \\ &\overset{\text{H}}{=} -\left(\lim_{s\to\infty}\left(-e^{-s}\right)\right) + 1 = 1\end{aligned}$$

(b) $\int_0^\infty t^{30}e^{-t}\,dt$ is equal to $\Gamma(31)$. From part (a) we found that $\Gamma(2) = 1$. We know that the Gamma function is a continuous version of the factorial, and on the positive integers $\Gamma(n) = (n-1)!$. Therefore $\int_0^\infty t^{30}e^{-t}\,dt = \Gamma(31) = 30!$.

Problem 8. Use the Comparison Theorem for improper integrals to determine the convergence of

$$\int_0^1 \frac{1}{\sqrt{x^3+x}}\,dx.$$

Solution. For all $x \in (0,1)$, $x^3 + x \geq x$. It follows that $\sqrt{x^3+x} \geq \sqrt{x}$ as well. Taking inverses, $\frac{1}{\sqrt{x^3+x}} \leq \frac{1}{\sqrt{x}}$ for all $x \in (0,1)$. Further, $\frac{1}{\sqrt{x^3+x}} \geq 0$ for all $x \in (0,1)$. Then, by the comparison test,

$$\begin{aligned}0 \leq \int_0^1 \frac{1}{\sqrt{x^3+x}}\,dx &\leq \int_0^1 \frac{1}{\sqrt{x}}\,dx \\ &= 2\sqrt{x}\Big|_0^1.\end{aligned}$$

Since this is finite, $\int_0^1 \frac{1}{\sqrt{x^3+x}}\,dx$ converges.

Problem 9. Consider the sequence given recursively by

$$a_1 = 2, \qquad\qquad a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right).$$

(a) Calculate a_2 and a_3.

(b) Assume that $\{a_n\}$ converges, and that $\lim_{x\to\infty} a_n = L$. Find L.

Solution. (a) We compute a_2 and a_3.

$$a_2 = \frac{1}{2}\left(a_1 + \frac{2}{a_1}\right) = \frac{1}{2}\left(2 + \frac{2}{2}\right) = \frac{3}{2}$$

$$a_3 = \frac{1}{2}\left(\frac{3}{2} + \frac{4}{3}\right) = \frac{17}{12}.$$

(b) Let $L = \lim_{n\to\infty} a_n$. Assuming $\{a_n\}$ converges, we can compute

$$L = \frac{1}{2}\left(L + \frac{2}{L}\right).$$

Solving this is equivalent to solving $L^2 = 2$. Since a_n is always positive, $L = \sqrt{2}$.

Problem 10. Determine whether the sequence given by

$$a_n = \frac{(\ln(n))^2}{n}$$

converges or diverges. If it converges, find its limit.

Solution.
Solution 1:

$$\lim_{n\to\infty} \frac{(\ln n)^2}{n} = \lim_{n\to\infty} \frac{2(\ln n)\cdot 1/n}{1} = \lim_{n\to\infty} \frac{2\ln n}{n} = \lim_{n\to\infty} \frac{2}{n} = 0.$$

Solution 2: Since $ln(n)$ and n are both positive for all $n > 2$, $\frac{(\ln(n))^2}{n}$ is also positive for all $n > 2$. Hence, it is bounded below by zero.

Let $f(x) = \sqrt{x} - (\ln x)^2$. We will show that $f(x)$ is positive for all $x > 1$, and this will imply $\frac{(\ln(n))^2}{n} \leq \frac{\sqrt{n}}{n}$ for all $n > 1$. $f(1) = 1 > 0$. If its derivative is always positive, then $f(x)$ will be increasing and $f(x)$ will positive for all $x > 1$. Its derivative $f'(x) = \frac{1}{2}\left(\frac{1}{\sqrt{x}}\right) - \frac{2\ln x}{x}$ is positive if and only if $\frac{1}{2}\sqrt{x} - 2\ln x$ is positive. This is positive for $x = 1$, and taking the second derivative shows that $f'(x)$ is increasing when $x > 1$. This implies $f(x) = \sqrt{x} - (\ln x)^2$ is always positive, so $\sqrt{x} \geq (\ln x)^2$. We have, for all $x > 1$, the inequalities

$$0 \leq \frac{(\ln(n))^2}{n} \leq \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}}.$$

Taking the limit as n approaches infinity,

$$0 \leq \lim_{n\to\infty} \frac{(\ln(n))^2}{n} \leq \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0.$$

By the Squeeze Theorem we conclude that $a_n = \frac{(\ln(n))^2}{n}$ converges to 0 as n approaches infinity.

Problem 11. (a) State the $\epsilon - N$ definition of $\lim_{n\to\infty} a_n = L$.

(b) Prove, using the definition asked for in part (a), that

$$\lim_{n\to\infty} \frac{1}{\ln n} = 0.$$

Solution. (a) A sequence $\{a_n\}$ is said to converge to a limit L if for every number $\epsilon > 0$, there exists a number $N \in \mathbb{N}$ such that for every $n > N$, $|a_n - L| < \epsilon$.

(b) Fix $\epsilon > 0$. Take $N \in \mathbb{N}$ such that $N > e^{1/\epsilon}$. Then, for all $n > N$,

$$\begin{aligned} \left|\frac{1}{\ln n}\right| &\leq \left|\frac{1}{\ln N}\right| \\ &\leq \left|\frac{1}{\ln e^{\frac{1}{\epsilon}}}\right| = \frac{1}{\left(\frac{1}{\epsilon}\right)} = \epsilon. \end{aligned}$$

The first inequality holds because $\ln n$ is increasing. We conclude $\lim_{n\to\infty} \frac{1}{\ln n} = 0$.