1. Using the definition of the limit, prove that

6 marks (a) $\lim_{n\to\infty} \frac{2^n}{1+2^n} = 1$.

If $a_n = \frac{2^n}{1+2^n}$, then $|a_n - 1| = \frac{1}{1+2^n}$

And if $\varepsilon > 0$ is a given positive number, then:

$|a_n - 1| < \varepsilon$ iff $\frac{1}{1+2^n} < \varepsilon$ iff $n > \frac{\ln(\frac{1}{\varepsilon} - 1)}{\ln(2)}$

No matter how small ε is, there is an integer, say N, which exceeds $\frac{\ln(\frac{1}{\varepsilon} - 1)}{\ln(2)}$.

Moreover if $n \geq N$, then $|a_n - 1| < \varepsilon$.

- since $n \geq N > \frac{\ln(\frac{1}{\varepsilon}) - 1}{\ln(2)} \Rightarrow |a_n - 1| < \varepsilon$

Thus, for every $\varepsilon > 0$ there is an integer N for which $|a_n - 1| < \varepsilon$ for all $n \geq N$.

Therefore $\lim_{n\to\infty} a_n = 1$

4 marks (b) $\lim_{n\to\infty} \sqrt{n^2 - 1} = \infty$.

If $M > 0$ is a given positive number, then:

$\sqrt{n^2 - 1} > M$ iff $n > \sqrt{M^2 + 1}$

No matter how large M is, there is an integer, say N, which exceeds $\sqrt{M^2 + 1}$

Moreover if $n \geq N$, then $\sqrt{n^2 - 1} \geq \sqrt{N^2 - 1} > M$

Thus, for every $M > 0$ there is an integer N for which $\sqrt{n^2 - 1} > M$ for all $n \geq N$

Therefore $\lim_{n\to\infty} \sqrt{n^2 - 1} = \infty$

2. Determine whether the following sequences are convergent or divergent. If a sequence is convergent, find its limit. Justify your answers.

6 marks (a) $a_n = \dfrac{\ln(n)}{\sqrt{n}}$.

Convergent:

$$\lim_{n\to\infty} \frac{\ln(n)}{\sqrt{n}} \overset{H}{=} \lim_{n\to\infty} \frac{1/n}{1/2\sqrt{n}}$$

$$= \lim_{n\to\infty} \frac{2}{\sqrt{n}}$$

$$= 0$$

6 marks (b) $a_n = \dfrac{\sin\left((-2)^n\right)}{n^3+1}$

Convergent:

$$|\sin((-2)^n)| \le 1 \implies |a_n| \le \frac{1}{n^3+1}$$

$$\text{i.e. } -\frac{1}{n^3+1} \le a_n \le \frac{1}{n^3+1}$$

$$\lim_{n\to\infty}\left(-\frac{1}{n^3+1}\right) = \lim_{n\to\infty} \frac{1}{n^3+1} = 0$$

$\therefore \lim_{n\to\infty} a_n = 0$ by Squeeze Theorem

6 marks (c) $a_n = (-1)^n \dfrac{n}{n+1}$

Divergent:

Even terms are $a_{2k} = (-1)^{2k} \cdot \dfrac{2k}{2k+1} = \dfrac{2k}{2k+1}$

$\implies a_{2k} \to 1$

Odd terms are $a_{2k+1} = (-1)^{2k+1} \cdot \dfrac{2k+1}{2k+2} = -\dfrac{2k+1}{2k+2}$

$\implies a_{2k+1} \to -1$

Even and odd terms approach different limits

$\implies$ sequence diverges

4 marks 3. (a) Give an example of a sequence which is bounded, but not convergent.

$a_n = (-1)^n$

bounded above by 1, below by -1

· divergent

4 marks (b) Given an example of a sequence which is not bounded above and not bounded below.

$a_n = n \cdot (-1)^n$

· even terms $\to \infty$ $\Rightarrow$ not bounded above

· odd terms $\to -\infty$ $\Rightarrow$ not bounded below

8 marks 4. Evaluate the sum of the infinite series $\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n^2+n}}$.

Hint: Consider the sequence of partial sums.

Partial Sums Are:

$$S_n = \sum_{k=1}^{n} \frac{\sqrt{k+1}-\sqrt{k}}{\sqrt{k^2+k}}$$

$$= \sum_{k=1}^{n} \frac{\sqrt{k+1}-\sqrt{k}}{\sqrt{k(k+1)}}$$

$$= \sum_{k=1}^{n} \left[\frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}}\right]$$

$$= \sum_{k=1}^{n} \frac{1}{\sqrt{k}} - \sum_{k=1}^{n} \frac{1}{\sqrt{k+1}}$$

$$= \left(1 + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{n}}\right) - \left(\frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}}\right)$$

$$= 1 - \frac{1}{\sqrt{n+1}}$$

Thus:

$$\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n^2+n}} = \lim_{n\to\infty} S_n$$

$$= \lim_{n\to\infty}\left[1 - \frac{1}{\sqrt{n+1}}\right]$$

$$= 1$$

The series converges, and its sum is equal to one.

8 marks 5. For what values of c does the series $\displaystyle\sum_{n=0}^{\infty} \frac{4^n + 6^n}{(2c)^n}$ converge? Justify your answer.

Let $a_n = \frac{4^n + 6^n}{(2c)^n}$. Note that $a_n = \left(\frac{2}{c}\right)^n + \left(\frac{3}{c}\right)^n$

Therefore $\sum a_n$ will converge if both $\sum \left(\frac{2}{c}\right)^n$ and $\sum \left(\frac{3}{c}\right)^n$ converge.

Each are geometric series, and $\sum \left(\frac{3}{c}\right)^n$ will converge provided $\left|\frac{3}{c}\right| < 1$, i.e. $|c| > 3$.
But if $|c| > 3$, then $\left|\frac{2}{c}\right| < 1$ as well

Therefore $\sum a_n$ converges if $|c| > 3$.

On the other hand, if $|c| \le 3$, then a_n does not converge to 0.
Indeed:
$$|a_n| = \left|\frac{4^n + 6^n}{(2c)^n}\right| = \frac{4^n + 6^n}{|2c|^n} \ge \frac{6^n}{|2c|^n} \ge 1 \quad \text{for all } n,$$
if $|c| \le 3$

Thus $|a_n|$ doesn't converge to 0, and hence a_n doesn't either.
Therefore $\sum a_n$ diverges by the Divergence Test.

We conclude that $\displaystyle\sum_{n=0}^{\infty} \frac{4^n + 6^n}{(2c)^n}$ converges if and only if $|c| > 3$ ($c > 3$ or $c < -3$)

8 marks 6. Determine whether the series $\sum_{n=1}^{\infty} \frac{\cos\left(\frac{1}{n}\right) + n^4}{(n^2+n+1)(n^3+1)}$ converges or diverges. Justify your answer.

Limit Comparison with $b_n = \frac{1}{n}$:

If $a_n = \frac{\cos(\frac{1}{n}) + n^4}{(n^2+n+1)(n^3+1)}$

Then:

$$\frac{a_n}{b_n} = \frac{n \cdot \cos(\frac{1}{n}) + n^5}{(n^2+n+1)(n^3+1)}$$

$$= \frac{\frac{\cos(\frac{1}{n})}{n^4} + 1}{(1+\frac{1}{n}+\frac{1}{n^2})(1+\frac{1}{n^3})}$$

Note that $\lim_{n\to\infty} \frac{\cos(\frac{1}{n})}{n^4} = 0$

$\cos(\frac{1}{n}) \to 1$, $n^4 \to \infty$

Thus

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{0+1}{(1+0+0)(1+0)} = 1$$

$0 < \lim_{n\to\infty} \frac{a_n}{b_n} < \infty$ And $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

$\sum_{n=1}^{\infty} \frac{\cos(\frac{1}{n}) + n^4}{(n^2+n+1)(n^3+1)}$ diverges by Limit Comparison.

8 marks 7. Suppose that a_n is a positive sequence (i.e. $a_n > 0$ for all n) with the property that

$$\sum_{n=1}^{N} a_n \leq 2011 - \frac{1}{N} \quad \text{for all} \quad N \geq 1.$$

Prove that the series $\sum_{n=1}^{\infty} a_n$ converges.

$$\sum_{n=1}^{\infty} a_n = \lim_{N \to \infty} \sum_{n=1}^{N} a_n$$

Thus the series converges if and only if the sequence $S_N = \sum_{n=1}^{N} a_n$ is a convergent sequence.

Since $a_n > 0$, S_N is increasing

$S_{N+1} - S_N = a_{N+1} > 0 \Rightarrow S_{N+1} > S_N$

Since $S_N \leq 2011 - \frac{1}{N} < 2011$, S_N is bounded

Thus S_N converges by the Monotonic Sequence Theorem.

Therefore $\sum_{n=1}^{\infty} a_n$ converges.

8 marks 8. Determine whether the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+4}$ is absolutely convergent, conditionally convergent or divergent. Justify your answer.

Let $a_n = (-1)^{n+1} \cdot \frac{n^2}{n^3+4}$

$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{n^2}{n^3+4}$, which diverges by Limit Comparison with $b_n = \frac{1}{n}$.

$\therefore$ the series is not absolutely convergent.

Now let $b_n = \frac{n^2}{n^3+4}$, so that $a_n = (-1)^{n+1} \cdot b_n$.

i) $\lim_{n \to \infty} b_n = 0$

ii) If $f(x) = \frac{x^2}{x^3+4}$, then $f'(x) = \frac{x(8-x^3)}{(x^3+4)^2}$

$\Rightarrow f'(x) < 0$ for $x > 2\sqrt[3]{2}$

$\Rightarrow b_n$ decreasing for $n \geq 3$

Therefore $\sum_{n=1}^{\infty} (-1)^{n+1} b_n$ converges by the Alternating Series Test

Thus $\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{n^2}{n^3+4}$ is conditionally convergent.

8 marks 9. Suppose that $\sum_{n=1}^{\infty} a_n = \frac{2}{3}$. Evaluate $\lim_{n\to\infty} \frac{1}{1+a_n}$.

$$\sum_{n=1}^{\infty} a_n = \frac{2}{3} \Rightarrow \sum_{n=1}^{\infty} a_n \text{ converges}$$

$$\Rightarrow \lim_{n\to\infty} a_n = 0$$

Therefore $\lim_{n\to\infty} \frac{1}{1+a_n} = \frac{1}{1+\lim_{n\to\infty} a_n}$ (since $f(x) = \frac{1}{1+x}$ is continuous at $x=0$)

$$= \frac{1}{1+0}$$

$$= 1$$

8 marks 10. Determine whether the series $\sum_{n=1}^{\infty} (-1)^n \frac{n!}{n^n}$ is absolutely convergent, conditionally convergent, or divergent. Justify your answer

You may use the fact that $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$.

Let $a_n = (-1)^n \cdot \frac{n!}{n^n}$

Then $|a_n| = \frac{n!}{n^n}$

$$\Rightarrow \left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!}$$

$$= \frac{(n+1)!}{n!} \cdot \frac{n^n}{(n+1)^{n+1}}$$

$$= (n+1) \cdot \frac{n^n}{(n+1)^n \cdot (n+1)}$$

$$= \left(\frac{n}{n+1}\right)^n$$

$$= \frac{1}{\left(1+\frac{1}{n}\right)^n}$$

Thus $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n}$

$$= \frac{1}{e}$$

$$< 1$$

$\therefore \sum_{n=1}^{\infty} (-1)^n \cdot \frac{n!}{n^n}$ is absolutely convergent by the Ratio Test.

8 marks 11. Determine whether the series $\sum_{n=1}^{\infty} \frac{(-1)^n n}{(\ln n)^n}$ is absolutely convergent, conditionally convergent, or divergent. Justify your answer.

Let $a_n = \frac{(-1)^n \cdot n}{(\ln n)^n}$

Then $|a_n| = \frac{n}{(\ln n)^n}$

$\Rightarrow \sqrt[n]{|a_n|} = \frac{n^{1/n}}{\ln n}$

Thus $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{n^{1/n}}{\ln n} = 0$

- $n^{1/n} \to 1$ since $\ln(n^{1/n}) = \frac{1}{n}\ln(n) \to 0$
- $\ln(n) \to \infty$

Since $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 0 < 1$, the series converges absolutely by the Root Test.