1. Consider the sequence given recursively by

$$a_1 = 2, \qquad a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right).$$

2 marks (a) Calculate a_2 and a_3.

$$a_2 = \frac{1}{2}\left(a_1 + \frac{2}{a_1}\right) = \frac{1}{2}\left(2 + \frac{2}{2}\right) = \frac{1}{2}\cdot 3 = 3/2$$

$$a_3 = \frac{1}{2}\left(a_2 + \frac{2}{a_2}\right) = \frac{1}{2}\left(\frac{3}{2} + \frac{2}{3/2}\right)$$
$$= \frac{1}{2}\left(\frac{3}{2} + 4/3\right)$$
$$= \frac{1}{2}\left(\frac{17}{6}\right)$$
$$= \frac{17}{12}$$

6 marks (b) Assume that $\{a_n\}$ converges, and that $\lim_{n\to\infty} a_n = L$. Find L.

$$\lim_{n\to\infty} a_n = L$$
so $\lim_{n\to\infty} a_{n+1} = L$

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{1}{2}\left(a_n + \frac{2}{a_n}\right)$$
$$= \frac{1}{2}\left[\lim_{n\to\infty} a_n + \frac{2}{\lim_{n\to\infty} a_n}\right]$$

$$L = \frac{1}{2}\left(L + \frac{2}{L}\right)$$
$$2L = L + \frac{2}{L}$$
$$L = \frac{2}{L}$$
$$L^2 = 2$$
$$L = \pm\sqrt{2}$$

But $a_n > 0$ for all n

so $L \geq 0$

Thus, $L = \sqrt{2}$

8 marks 4. Determine whether the series

$$\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right)$$

converges or diverges. If it converges, find its sum.

Note that $\ln\left(\frac{n+1}{n}\right) = \ln(n+1) - \ln n$

so
$$\begin{aligned} S_n &= \sum_{k=1}^{n} \ln\left(\frac{k+1}{k}\right) \\ &= \sum_{k=1}^{n} [\ln(k+1) - \ln k] \\ &= (\ln 2 - \ln 1) + (\ln 3 - \ln 2) + \cdots \\ &\qquad + \ln(n+1) - \ln n \\ &= \ln 3 - \ln(n+1) \end{aligned}$$

so
$$\begin{aligned} \lim_{n\to\infty} S_n &= \lim_{n\to\infty} [\ln 3 - \ln(n+1)] \\ &= \ln 3 - (-\infty) \\ &= \infty \end{aligned}$$

Thus, $\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right)$ diverges.

2 marks 5. (a) State the Monotone Sequence Theorem.

Every bounded, monotonic sequence is convergent.

8 marks (b) Suppose $a_n > 0$ for $n = 1, 2, \ldots$, and that

$$s_n = \sum_{k=1}^{n} a_k \quad < \quad 2 - \frac{1}{n}$$

for $n > 1$. Prove that $\sum_{n=1}^{\infty} a_n$ converges.

Proof

$a_n > 0$ for $n = 1, 2 \ldots$

$s_{n+1} = s_n + a_n > s_n$ for $n = 1, 2 \ldots$

Thus $\{s_n\}$ is increasing

Also $s_1 = 2 - 1 = 1$ and $s_n < 2$

so $1 \leq s_n < 2$ for $n = 1, 2, \ldots$

Thus $\{s_n\}$ is bounded

Hence $\{s_n\}$ converges by the Monotone Sequence Theorem.

i.e., Σa_n converges.

8 marks 6. Determine whether the series

$$\sum_{n=1}^{\infty} \frac{n^3+n^2-3}{\sqrt{n^9+7n^3+3n}}$$

converges or diverges. Justify your answer using an appropriate test.

Let $a_n = \dfrac{n^3+n^2-3}{\sqrt{n^9+7n^3+3n}}$ and $b_n = \dfrac{n^3}{\sqrt{n^9}} = \dfrac{n^3}{n^{9/2}}$

$$= \frac{1}{n^{3/2}}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\dfrac{n^3+n^2-3}{\sqrt{n^9+7n^3+3n}}}{1/n^{3/2}}$$

$$= \lim_{n\to\infty} \frac{n^{3/2}(n^3+n^2-3)}{\sqrt{n^9+7n^3+3n}}$$

$$= \lim_{n\to\infty} \frac{n^{3/2}(n^3+n^2-3)/n^{9/2}}{\sqrt{n^9+7n^3+3n}\,/n^{9/2}}$$

$$= \lim_{n\to\infty} \frac{1+1/n-3/n^3}{\sqrt{1+7/n^6+3/n^8}}$$

$$= \frac{1+0-0}{\sqrt{1+0+0}}$$

$$= 1$$

$\sum b_n = \sum \frac{1}{n^{3/2}}$ converges (p-series, $p = 3/2 > 1$)

Thus, $\sum a_n$ converges by the Limit Comparison Test.

8 marks 7. Determine whether the series

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$$

converges or diverges. Justify your answer using an appropriate test.

Let $a_n = \frac{1}{n(\ln n)^2}$ and $f(x) = \frac{1}{x(\ln x)^2}$

f is positive, continuous, and decreasing on $[2,\infty)$.

$$\int_2^{\infty} f(x)\,dx = \lim_{t\to\infty}\int_2^t \frac{1}{x(\ln x)^2}\,dx.$$

$\int \frac{1}{x(\ln x)^2}\,dx$ | Let $u = \ln x$, $du = \frac{1}{x}\,dx$

$= \int \frac{1}{u^2}\,du$

$= \frac{u^{-1}}{-1} + C$

$= -\frac{1}{\ln x} + C$

So $\int_2^t \frac{1}{x(\ln x)^2}\,dx = -\frac{1}{\ln x}\Big|_2^t = \frac{-1}{\ln t} + \frac{1}{\ln 2}$

Thus, $\lim_{t\to\infty}\int_2^t \frac{1}{x(\ln x)^2}\,dx = \lim_{t\to\infty} \frac{-1}{\ln t} + \frac{1}{\ln 2}$

$= 0 + \frac{1}{\ln 2}$

Hence, $\int_2^{\infty} f(x)\,dx$ converges

Thus, $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ converges by the Integral Test.

8 marks 8. Determine whether the series

$$\sum_{n=1}^{\infty} \frac{n+4^n}{3^n}$$

converges or diverges. Justify your answer using an appropriate test.

3 solutions
There are others

Let $a_n = \dfrac{n+4^n}{3^n}$, $b_n = \dfrac{4^n}{3^n} = \left(\dfrac{4}{3}\right)^n$

$a_n > b_n$

$\sum b_n = \sum (4/3)^n$ is a geometric series with $r = 4/3$ so $|r| = 4/3 > 1$.

Thus, $\sum b_n$ diverges.

Hence, $\sum a_n$ diverges by the Comparison Test

OR

$$\frac{n+4^n}{3^n} = \frac{n}{3^n} + \left(\frac{4}{3}\right)^n$$

$$\to 0 + \infty = \infty \neq 0$$

Thus, $\sum \dfrac{n+4^n}{3^n}$ diverges by the Divergence Test

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OR Let $a_n = \dfrac{n+4^n}{3^n}$

Then $$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\dfrac{n+1+4^{n+1}}{3^{n+1}}}{\dfrac{n+4^n}{3^n}} = \frac{n+1+4^{n+1}}{3\,(n+4^n)}$$

$$= \frac{\dfrac{n+1}{4^n} + 4}{3\left(\dfrac{n}{4^n}+1\right)} \to \frac{0+4}{3(0+1)} = 4/3$$

Thus, $\sum a_n$ diverges by the Ratio Test.

8 marks 9. Determine whether the series

$$\sum_{n=1}^{\infty} \frac{n^{4n}}{(n!)^n}$$

converges or diverges. Justify your answer using an appropriate test.

Let $a_n = \dfrac{n^{4n}}{(n!)^n} = \left(\dfrac{n^4}{n!}\right)^n$

$$\sqrt[n]{|a_n|} = \sqrt[n]{\left(\frac{n^4}{n!}\right)^n} = \frac{n^4}{n!}$$

$$= \frac{n \cdot n \cdot n \cdot n}{n \cdot (n-1)(n-2)(n-3)(n-4)\cdots 1}$$

$$= \frac{n}{n-1} \cdot \frac{n}{n-2} \cdot \frac{n}{n-3} \cdot \frac{1}{(n-4)(n-5)\cdots 1}$$

$$\to 1 \cdot 1 \cdot 1 \cdot 0 = 0$$

Thus $\sum a_n$ converges by the Root Test

8 *marks* 10. Approximate the sum of the (convergent) series

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{(2n)!}$$

to within an error of 0.01. Leave your answer as the sum of fractions.

Let $S = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(2n)!}$

$S_n = \sum_{k=1}^{n} (-1)^{k-1} \frac{1}{(2k)!}$

$b_n = \frac{1}{(2n)!}$

$\{b_n\}$ is decreasing and $\lim b_n = 0$ so the Alternating Series Estimation Theorem applies, so $|S - S_n| \leq b_{n+1}$

For $n=1$, $b_{1+1} = b_2 = \frac{1}{4!} = \frac{1}{24} > \frac{1}{100}$

For $n=2$, $b_{2+1} = b_3 = \frac{1}{6!} = \frac{1}{720} < \frac{1}{100}$

so for $n=2$, $|S - S_n| < \frac{1}{100}$

$S_2 = \frac{1}{2!} - \frac{1}{4!}$

$= \frac{1}{2} - \frac{1}{24}$

2 marks 11. (a) Define what it means for $\sum_{n=1}^{\infty} a_n$ to be conditionally convergent.

$\sum a_n$ is conditionally convergent if $\sum a_n$ is convergent, but not absolutely convergent.

8 marks (b) Determine whether the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{n}$$

is conditionally convergent, absolutely convergent or divergent. Justify your answer using one or more appropriate tests.

Let $b_n = \frac{\ln n}{n}$ and $f(x) = \frac{\ln x}{x}$

$f'(x) = \frac{x\left(\frac{1}{x}\right) - \ln x}{x^2} = \frac{1-\ln x}{x^2} < 0$ if $x > e$.

Thus, f is decreasing on $[e, \infty)$

so $\{b_n\}$ is decreasing for $n \geq 3$

$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{\ln n}{n} \stackrel{H}{=} \lim_{n\to\infty} \frac{1/n}{1} = 0$

Thus $\sum (-1)^n \frac{\ln n}{n}$ converges by the Alternating Series Test.

Let $a_n = (-1)^n \frac{\ln n}{n}$. Then $|a_n| = \frac{\ln n}{n}$. Let $b_n = 1/n$

$|a_n| = \frac{\ln n}{n} > \frac{1}{n} = b_n$

$\sum b_n$ diverges (Harmonic Series)

Thus, $\sum |a_n|$ diverges by the Comparison Test

i.e., $\sum a_n$ is not absolutely convergent.

Hence $\sum a_n$ is conditionally convergent.

8 marks 12. Determine whether the series

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(n!)^2}{(2n)!}$$

is conditionally convergent, absolutely convergent or divergent. Justify your answer using one or more appropriate tests.

Let $a_n = (-1)^{n+1} \dfrac{(n!)^2}{(2n)!}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{n+2}[(n+1)!]^2 / (2n+2)!}{(-1)^{n+1}(n!)^2 / (2n)!}\right|$$

$$= \frac{[(n+1)!]^2}{(n!)^2} \cdot \frac{(2n)!}{(2n+2)!}$$

$$= \left(\frac{n+1}{n}\right)^2 \cdot \frac{1}{(2n+1)(2n+2)}$$

$$\to 1^2 \cdot 0 = 0$$

Thus, $\sum a_n$ is absolutely convergent by the Ratio Test.