2010 Final Exam

1. Find the radius and interval of convergence for the power series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\,(x-4)^n}{n\,3^n}\,.$$

2. Let $f(x) = x\sin(x^2)$.

   (a) Find the Maclaurin series for $f(x)$.

   (b) Find $f^{(31)}(0)$.

   (c) Find $f^{(32)}(0)$.

3. Find the Taylor series for $f(x) = \ln(x)$, centred at $a = 4$, and determine its interval of convergence.

4. Find the Maclaurin series for $f(x) = \dfrac{x}{1 - x^2}$.

5. (a) Find the Maclaurin series for $\dfrac{\tan^{-1}(x)}{x}$.

(b) Use your answer to part (a) to express the definite integral $\displaystyle\int_0^1 \frac{\tan^{-1}(x)}{x}\,dx$ as the sum of an infinite series.

(c) According to the Alternating Series Estimation Theorem, what is the smallest number of terms required to approximate the integral in (b) with an error magnitude no larger than 10^{-4}?

6. (a) Find the first four non-zero terms in the Maclaurin series for $\dfrac{1}{\sqrt{1-x^2}}$.

(b) Use your answer from (a) to find the first four non-zero terms in the Maclaurin series for $f(x) = \sin^{-1} x$.

7. Let C be the curve given parametrically by $x = t - \sin t$, $y = 1 - \cos t$, for $-\pi \leq t \leq \pi$.

   (a) Find $\frac{dy}{dx}$ in terms of t.

   (b) Find an equation for the line which is tangent to C at the point where $t = \frac{\pi}{3}$.

8. Let γ be the curve given parametrically by $x = \frac{t^2}{2} - t$, $y = \frac{4}{3}t^{3/2}$, for $0 \leq t \leq A$, where $A > 0$ is some number. Find A so that the length of γ equals 4.

9. Find the length of the polar curve given by $r = \sin^2\left(\frac{\theta}{2}\right)$ for $0 \leq \theta \leq \pi$.

10. Prove: If $a_n > 0$ for all $n \geq 1$ and $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$ converges.

11. Evaluate $\int \ln\left(x^2 - 1\right)\, dx$.