AREA AND RIEMANN SUMS

These lecture notes are designed to provide supplementary material to Stewart, "Single Variable Calculus, Seventh Edition, with Early Transcendentals". This is far from a complete, or even rigorous treatment of set theory. Such a treatment would lead us too far astray. This is just enough to get us going in mathematics.

As this is an enriched course, some of the material taught is for the interest of the student and will not appear on exams. This matherial is differentiated from examinable material by a blue font colour.

1. Upper and Lower Sums

Definition 1.1. A *partion of an interval* $[a, b]$ *of length* n is an increasing sequence of points in $[a, b]$ of the form

$$a = t_0 < t_1 < t_2 < \ldots < t_n = b.$$

Example 1.1. The *standard partition of length* n of an interval is that partition that cuts the interval into n pieces. So the standard partition of length 4 of the interval $[0, 1]$ is

$$0 < 0.25 < 0.5 < 1.$$

We will denote the standard partition of length n by $\Gamma_n([a, b])$ or when the interval is understood, Γ_n.

Definition 1.2. Suppose that f is a bounded function on $[a, b]$, ie there is a real number $M > 0$ so that

$$-M < f(x) < M.$$

Let P be the partition of $[a, b]$ given by

$$a = t_0 < t_1 < t_2 < \ldots < t_n = b.$$

Set

$$\begin{aligned} m_i &= \inf\{f(x) | t_{i-1} \le x \le t_i\} \\ M_i &= \sup\{f(x) | t_{i-1} \le x \le t_i\}. \end{aligned}$$

The *lower sum* of f for P is defined to be

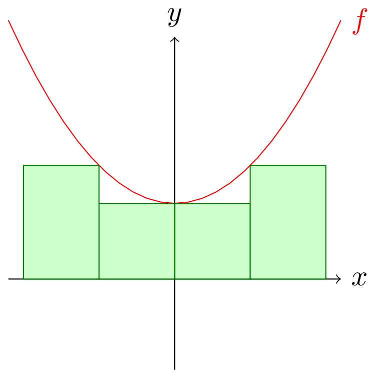
$$L(f, P) = \sum_{i=1}^{n} m_i(t_i - t_{i-1}).$$

The *upper sum* of f for P is defined to be

$$U(f, P) = \sum_{i=1}^{n} M_i(t_i - t_{i-1}).$$

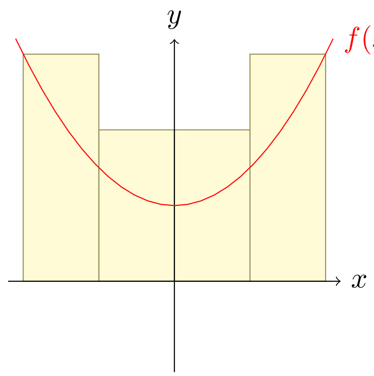
A picture of a lower sum.

The lower sum $L(\Gamma_n[-1,1], x^2+1)$ computes the area of the green region pictured below.



A picture of an upper sum

The upper sum $U(\Gamma_n[-1,1], x^2+1)$ computes the area of the yellow region pictured below.



Example 1.2. Consider the partition $P = (t_0 = -2 < t_1 = 2 < t_2 = 3 < t_3 = 6)$ of the interval $[-2, 6]$. Let us try to calculate the lower and upper sums of $f(x) = e^x$ with respect to this partition. The function $f(x) = e^x$ is increasing as its derivative will always be positive. So

$$\begin{aligned} e^{t_{i-1}} = m_i &= \inf\{e^x | t_{i-1} \leq x \leq t_i\} \\ e^{t_i} = M_i &= \sup\{f(x) | t_{i-1} \leq x \leq t_i\}. \end{aligned}$$

So we get

$$\begin{aligned} L(e^x, P) &= 4.e^{-2} + e^2 + 3.e^3 \quad \text{and} \\ U(e^x, P) &= 4.e^2 + e^3 + 3.e^6. \end{aligned}$$

2. The definite integral

Let f be a bounded function on $[a, b]$. We define the *lower integral* of f on $[a, b]$ be

$$\underline{\int_a^b} f(x)dx = \sup\{L(f, P) | P \text{ a partition of } [a, b]\}$$

Note : that the supremum is over all possible partitions of the interval $[a, b]$. This exists as a number as $L(f, P) \leq M(b-a)$ where M is an upper bound for $f(x)$.

Similarly, we define the upper integral by

$$\overline{\int_a^b} f(x)dx = \inf\{U(f,P)|P \text{ a partition of } [a,b]\}$$

It is convenient to use standard partitions when calculating upper and lower integrals. We have

Lemma 2.1.

$$\begin{aligned}
\underline{\int_a^b} f(x)dx &= \lim_{n\to\infty} L(f, \Gamma_n[a,b]) \\
\overline{\int_a^b} f(x)dx &= \lim_{n\to\infty} U(f, \Gamma_n[a,b])
\end{aligned}$$

Proof. Ommitted for now. □

Example 2.1. Lets calculate the integrals

$$\underline{\int_0^1} x^2 dx \quad \text{and} \quad \overline{\int_0^1} x^2 dx.$$

We will make use of the formulation in (2.1). The function $y = x^2$ is increasing on the interval $[0, 1]$ so the M_i's will occurr at the right endpoints of subintervals in the partition and the m_i's at the left endpoints. The standard partition $\Gamma_n[0, 1]$ looks like

$$0 = t_0 < \frac{1}{n} < \frac{2}{n} < \ldots \frac{i}{n} < \frac{i+1}{n} < \ldots < \frac{n}{n} = n$$

So we obtain

$$\begin{aligned}
U(f, \Gamma_n[0,1]) &= \sum_{i=1}^{n} \frac{1}{n}(\frac{i}{n})^2 \\
&= \frac{1}{n^3}\sum_{i=1}^{n} i^2 \\
&= \frac{1}{n^3}\frac{n(n+1)(2n+1)}{6}.
\end{aligned}$$

The last line follows from the formula in the previous section on sums of squares. So

$$\begin{aligned}
\overline{\int_0^1} x^2 dx &= \lim_{n\to\infty} \frac{1}{n^3}\frac{n(n+1)(2n+1)}{6} \\
&= \lim_{n\to\infty} \frac{(1+1/n)(2+1/n))}{6} \\
&= 1/3.
\end{aligned}$$

Now for the lower integral. We obtain

$$\begin{aligned} L(f, \Gamma_n[0,1]) &= \sum_{i=0}^{n-1} \frac{1}{n} (\frac{i}{n})^2 \\ &= \frac{1}{n^3} \sum_{i=1}^{n-1} i^2 \\ &= \frac{1}{n^3} \frac{n(n-1)(2n-1)}{6}. \end{aligned}$$

The last line follows from the formula in the previous section on sums of squares. So

$$\begin{aligned} \underline{\int}_0^1 x^2 dx &= \lim_{n\to\infty} \frac{1}{n^3} \frac{n(n-1)(2n-1)}{6} \\ &= \lim_{n\to\infty} \frac{(1-1/n)(2-1/n))}{6} \\ &= 1/3. \end{aligned}$$

Hence the lower and upper integrals are the same. Such a function is termed *integrable*.

Example 2.2. Lets consider the function defined by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ rational} \\ 0 & \text{otherwise.} \end{cases}$$

Every interval of positive length contains both rational and irrational numbers. Hence $M_i = 1$ and $m_i = 0$, always. Running through the definitions we find

$$\underline{\int}_0^1 f(x)dx = 0 \quad \text{and} \quad \overline{\int}_0^1 f(x)dx = 1.$$

Theorem 2.2. *i)*

$$\overline{\int}_a^b (f(x) + g(x))dx = \overline{\int}_a^b f(x)dx + \overline{\int}_a^b g(x)dx$$

ii)

$$\underline{\int}_a^b (f(x) + g(x))dx = \underline{\int}_a^b f(x)dx + \underline{\int}_a^b g(x)dx$$

iii) If $c > 0$ then

$$\overline{\int}_a^b cf(x)dx = c\overline{\int}_a^b f(x)dx$$

iv) If $c > 0$ then

$$\underline{\int}_a^b cf(x)dx = c\underline{\int}_a^b f(x)dx$$

v)

$$\underline{\int}_a^b -f(x)dx = -\overline{\int}_a^b f(x)dx$$

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Proof. These are mostly straightforward. We will only write down proofs for a couple of them and the rest will be left as exercises.
i) For an arbitrary partition $P = (t_0 < .. < t_n)$ we have that

$$\sup\{f(x)|x \in [t_{i-1}, t_i]\} + \sup\{g(x)|x \in [t_{i-1}, t_i]\} = \sup\{f(x) + g(x)|x \in [t_{i-1}, t_i]\}.$$

Hence $U(f + g, P) = U(f, P) + U(g, P)$. The result follows.
v) This follows from the fact that

$$-\sup\{f(x)|x \in [t_{i-1}, t_i]\} = \inf\{-f(x)|x \in [t_{i-1}, t_i]\}$$

□

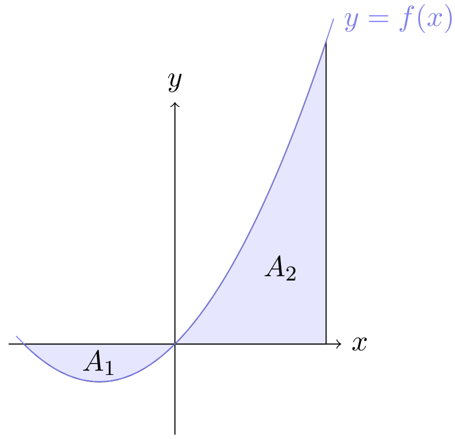
Definition 2.3. We say that a function is *integrable* if

$$\underline{\int_a^b} f(x)dx = \overline{\int_a^b} f(x)dx.$$

In this case we define its integral to be the common value and write

$$\int_a^b f(x)dx = \underline{\int_a^b} f(x)dx = \overline{\int_a^b} f(x)dx.$$

Intuitively, the definite integral is a signed area.



So $\int_a^b f(x)dx = A_2 - A_1$

Theorem 2.4. *If f is continuous on $[a, b]$ then it is integrable on $[a, b]$.*

Proof. Ommitted for now. □

3. Calculating areas

Given a $f(x) \geq 0$ on $[a, b]$. If $f(x)$ is integrable on $[a, b]$ we define the area under $y = f(x)$ to be

$$\int_a^b f(x)dx.$$

In the next section we will introduce a quick method for computing such areas for nice functions. For now, lets proceed with a hands on approach. In order to do this the following sums from the previous section will be useful. (You do not need to remember these, they will be given to you on the final if needed.) Recall

$$\begin{aligned}\sum_{i=1}^{n} i &= \frac{n(n+1)}{2}\\ \sum_{i=1}^{n} i^2 &= \frac{n(n+1)(2n+1)}{6}\\ \sum_{i=1}^{n} i^3 &= \left(\frac{n(n+1)}{2}\right)^2\end{aligned}$$

Example 3.1. We will work out the following example in class :

$$\int_2^4 (x^2+2x)dx = \frac{56}{3}+8$$

Example 3.2. We will work out the following example in class :

$$\int_0^2 (x^3+x)dx = 18$$

4. The definite integral and the fundamental theorem of calculus part II

The fundamental theorem of calculus has two parts, rougly inverse to each other. We will meet part I later, here is part II.

Theorem 4.1. *Suppose that $f(x)$ and $g(x)$ are continuous on $[a,b]$ and $f(x)=g'(x)$ on (a,b). Then*

$$\int_a^b f(x)dx = g(b)-g(a).$$

Such a g is called an *antiderivative* for f.

Proof. Consider a partition $P=(t_0=a<t_1<t_2<\ldots<t_n=b)$. Let M_i be the supremum of f on $[t_{i-1},t_i]$. By the mean value theorem, applied to $g(x)$, we have

$$\frac{g(t_i)-g(t_{i-1})}{t_i-t_{i-1}} \leq M_i.$$

Hence

$$\begin{aligned}U(f,P) &= \sum_{i=1}^{n} M_i(t_i-t_{i-1})\\ &\geq \sum_{i=1}^{n} g(t_i)-g(t_{i-1})\\ &= g(b)-g(a).\end{aligned}$$

Similarly we have $L(f,P)\leq g(b)-g(a)$. Hence

$$\underline{\int_a^b} f(x)dx \leq g(b)-g(a) \leq \overline{\int_a^b} f(x)dx.$$

But by (2.4) the above chain must consist only of equalities. $\square$

Example 4.1. The examples (3.1) and (3.2) can be computed quickly with this theorem. In fact, the fuctions $x^3/3 + x^2$ and $x^4/4 + x^2/2$ are antiderivatives.

5. Exercises

5.1. Describe the partition $\Gamma_{10}[2.1, 3.1]$

5.2. Consider the partition $P = (t_0 = -3 < t_1 = 4 < t_3 = 8 < t_4 = 12)$ of $[-3, 12]$. Calculate the following upper/lower sums :
  (a) $L(e^x, P)$
  (b) $U(x^2 - 2x, P)$
  (c) $L(-2x^2 - x, P)$
  (d) $U(-2x^2 - x, P)$

5.3. Fill in the details for example 2.2.

5.4. Show that the function defined below is not integrable on $[0, 1]$

$$f(x) = \begin{cases} x^2 & \text{if } x \text{ rational} \\ 0 & \text{otherwise.} \end{cases}$$

Compute, directly from definitions, the upper and lower integrals

$$\underline{\int_0^1} f(x)dx \quad \overline{\int_0^1} f(x)dx$$

5.5. Prove part iv of theorem 2.2.

5.6. Calculate the following integral in two ways, using and not using the fundamental theorem of calculus.

$$\int_{-1}^{2} (3x^2 - x + 1)dx.$$

5.7. Suppose that the functions $f(x)$ and $g(x)$ are integrable on $[a, b]$. Use theorem 2.2 to show that

$$\int_a^b (cf(x) + dg(x))dx = c\int_a^b f(x)dx + d\int_a^b g(x)dx,$$

where c and d are some real numbers.