THE EXPONENTIAL FUNCTION

These lecture notes are designed to provide supplementary material to Stewart, "Single Variable Calculus, Sixth Edition, with Early Transcendentals". This is far from a complete, or even rigorous treatment of set theory. Such a treatment would lead us too far astray. This is just enough to get us going in mathematics.

As this is an enriched course, some of the material taught is for the interest of the student and will not appear on exams. This matherial is differentiated from examinable material by a blue font colour.

1. Functions

Recall from the notes on sets :

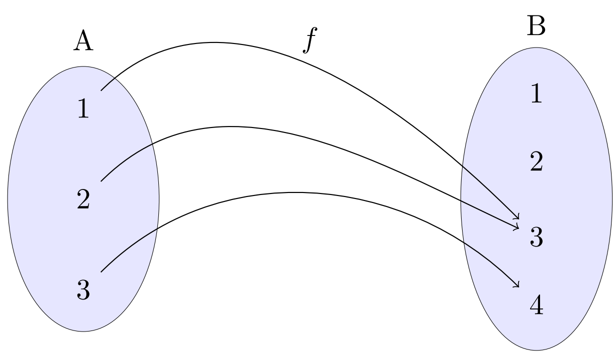
Definition 1.1. Let D and C be two sets. A *function from D to C* denoted $f : D \to C$ is a rule that assigns to each element of D and element of C. The set D is called the *domain* of the function. The set C is called the *codomain* of the function. (Note : C is not in general the range.)

This course is mostly concerned with functions $f : D \to \mathbb{R}$ where D is a subset of $\mathbb{R}$.

Example 1.1. Consider the sets $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4\}$. We define a function

$$f : A \to B$$

with $f(1) = 3,\ f(2) = 3,\ f(3) = 3.$ Pictorially :



In this course we will mostly be concerned with functions whose domains and codomains are subsets of $\mathbb{R}$. The *range* of a function $f : D \to C$ is the following subset of the codomain :

$$\text{range}(f) = \{c \in C | c = f(d) \text{ for some } d \in D\}$$

2. The Exponential Function

In this section we will construct a family of functions whose domain and codomain are $\mathbb{R}$

Proposition 2.1. *Let $a \in \mathbb{R}$ with $a > 0$. Suppose that $n > 0$ is an integer. Then there is a unique $y \in \mathbb{R}$ with $y > 0$ and $y^n = a$.*

Proof. For simplicity we will assume $n = 2$. The general case is similar, only the algebra is a little harder. Feel free to discuss this with the instructor.

We need to show two things, that y exists and it is unique.

Existence Consider the set $S = \{z > 0 | z^2 \le a\}$. This set is bounded above by. (Why? See the exercises) Hence is has a supremum. Let y be the supremum. So for every $\epsilon > 0$ the number $y - \epsilon$ cannot be an upper bound for S. Hence

$$\begin{aligned} a &> (y-\epsilon)^2 \\ &= y^2 - \epsilon(2y+\epsilon). \end{aligned}$$

Hence

$$a + \epsilon(2y+\epsilon) > y^2.$$

As $\epsilon > 0$ can be made arbitrarily small we must have $a \ge y^2$.

To complete the existence part of the proof we need to show that $a > y^2$ is not possible. We prove this by contradiction, that is we assume that $a > y^2$ and show that this implies something absurd. So suppose that $a > y^2$. Let $0 < \delta = a - y^2$. Now we can choose $0 < \lambda$ so that $\lambda(2y + \lambda^2) < \delta$. (why? See exercises). Then

$$\begin{aligned} (y+\lambda)^2 &= y^2 + \lambda(2y+\lambda) \\ &< y^2 + \delta \\ &< a. \end{aligned}$$

So we conclude that $y + \lambda \in S$. But this is a contradiction as y is supposed to be a sup. Hence we must have $y^2 =$.

Uniqueness Let $y' > 0$ be another such solution. We show that y' must be $\sup S$ to prove uniqueness. First we need to show that y' is an upper bound. Suppose that $z^2 \le a$ but $y' < z$. Then

$$\begin{aligned} {y'}^2 &< z^2 \\ &\le a \end{aligned}$$

which is a contradiction. Hence y' is an upper bound. A similar argument shows that it is smaller than any other upper bound. Hence $y' = \sup S$, proving uniqueness. □

Let $a > 0$. We denote by $\sqrt[n]{a}$ or $a^{1/n}$ the unique positive solution to the equation

$$x^n = a.$$

If $p = m/n$ is a rational number, we denote by a^p the number $(\sqrt[n]{a})^m$. Note that m maybe negative here but $n > 0$.

Lemma 2.2. *The above definition does not depend on choice of the representation $p = m/n$. In other words, if $m/n = m'/n'$ then*

$$(\sqrt[n]{a})^m = (\sqrt[n']{a})^{m'}$$

Proof. We have, by definiton, $((\sqrt[n]{a})^m)^n = a^m$. Hence

$$
\begin{aligned}
((\sqrt[n]{a})^m)^{n'} &= (\sqrt[n]{a})^{mn'} \\
&= (\sqrt[n]{a})^{m'n} \\
&\quad \text{as } m'n = mn' \\
&= a^{m'}.
\end{aligned}
$$

So $(\sqrt[n]{a})^m$ is a positive solution to the equation $x^{n'} = a^{m'}$. But $(\sqrt[n']{a})^{m'}$ is also a positive solution. By the uniqueness part of 2.1 we obtain

$$(\sqrt[n]{a})^m = (\sqrt[n']{a})^{m'}.$$

□

Lemma 2.3. *For two rational numbers p and q we have*

$$a^{p+q} = a^p a^q.$$

Proof. An exercise. □

Suppose that $a > 1$. We define the the exponential function $f(x) = a^x$ by the formula

$$f(x) = \sup\{a^p | p \in \mathbb{Q}\ p < x\}.$$

If $0 < a < 1$ we define $a^x = (1/a)^{-x}$. This makes sense as $0 < a < 1$ implies $a < 1$.

Proposition 2.4. (i) $a^{x+y} = a^x a^y$.
(ii) *If $a > 1$ the exponential function a^x is an increasing function, that is $x < y$ implies $a^x < a^y$.*
(iii) *If $0 < a < 1$ the exponential function a^x is a decreasing function, that is $x < y$ implies $a^x > a^y$.*
(iv) *If $a = 1$ the function is constant.*
(v) *If $a \neq 1$ and $a > 0$ then the range of the exponential function is $(0, \infty) = \{x \in \mathbb{R} | x > 0\}$*

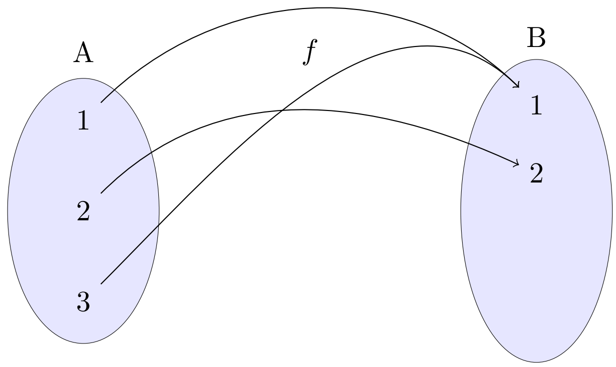
Proof. Omitted. (Somewhat involved, especially part (v). Please feel free to discuss with the instructor) □

3. One-to-one and onto functions

Consider a function $f : D \to C$.

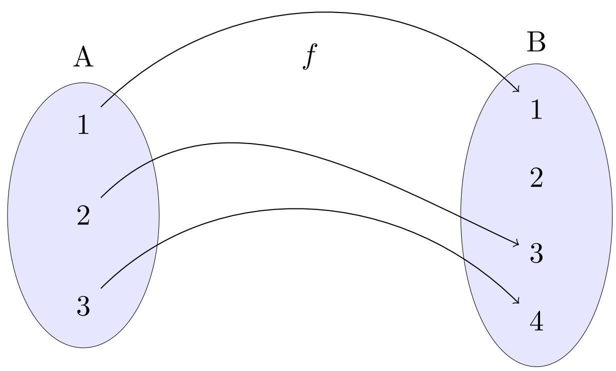
We say that f *onto* if $\text{range}(f) = C$. In other words for every $c \in C$ we can find a $d \in D$ with $f(d) = c$.

Example 3.1. Here is an example of an onto function.



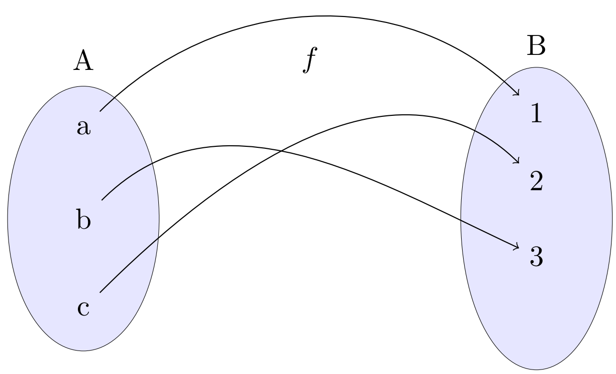
We say that f is *one-to-one* if $f(x) = f(y)$ implies $x = y$. Equivalently, if $x \neq y$ then $f(x) \neq f(y)$.

Example 3.2. Here is an example of a one-to-one function.



A *bijection* is a function that is both onto and one-to-one.

Example 3.3. Here is an example of a one-to-one function.



Suppose that $f : D \to C$ is a bijection. Then we can form a new function $f^{-1} : C \to D$, called *the inverse function*. It is defined by the rule

$$f^{-1}(y) = x \quad \text{if and only if} \quad f(x) = y.$$

Lemma 3.1. *Suppose $a > 0$ and $a \neq 1$ then the function a^x is one-to-one.*

Proof. This follows directly from 2.4 parts (ii) and (iii) □

The exponential function a^x viewed as a function $\mathbb{R} \to \mathbb{R}$ is not onto. However if we view it as a function $\mathbb{R} \to (0, \infty)$ we obtain a bijection. Hence there is an inverse function, call the logarithm and written $\log_a(x)$. In summary :

$$\log_a(x) = y \quad \text{if and only if} \quad a^y = x.$$

4. Exercises

4.1. Prove 2.3.
4.2. Let $a > 0$ be a real number. Show that the set $\{z \in \mathbb{R} | z^2 \leq a\}$ has an upper bound and hence a supremum.
4.3. Let $\delta > 0$ and y a positive real number. Show that there is a $\lambda > 0$ with $\lambda(2y + \lambda) < \delta$.
4.4. Suppose $a > 1$. Using only the definition of of the exponential function that if $x < y$ then $a^x \leq a^y$. (Do not use 2.4)
4.5. What is $\log_{10}(100)$?
4.6. Which of the following functions give bijections $[0, 1] \to [0, 1]$?

$$f(x) = x \quad f(x) = x^2 \quad f(x) = x/2 \quad f(x) = \sqrt{x}.$$