LIMITS : PART1

These lecture notes are designed to provide supplementary material to Stewart, ”Single Variable Calculus, Seventh Edition, with Early Transcendentals”. This is far from a complete, or even rigorous treatment of set theory. Such a treatment would lead us too far astray. This is just enough to get us going in mathematics.

As this is an enriched course, some of the material taught is for the interest of the student and will not appear on exams. This matherial is differentiated from examinable material by a blue font colour.

1. Getting started

We wish to give meaning to the notion of limit. The text gives the following defintion :

Definition 1.1. Let $f(x)$ be a real valued function defined on some interval containing a. We say the the *limit of* $f(x)$*, as* x *approaches* a*, equals* L and write

$$\lim_{x \to a} f(x) = L$$

if we can make the values of $f(x)$ arbitrarily close to L by taking x sufficently close to a but not equal to a.

The short comings of this definition will be discussed in the next section. Lets use it now to see what a limit means.

Example 1.1. What is $\lim_{x \to 4} x^2$?

Example 1.2. Consider

$$f(x) = \begin{cases} x & x > 0 \\ 10 & x = 0 \\ x & x < 0 \end{cases}$$

What is $\lim_{x \to 0} f(x)$?

Example 1.3. Consider

$$f(x) = \begin{cases} x & x > 0 \\ 10 & x = 0 \\ x - 1 & x < 0 \end{cases}$$

What is $\lim_{x \to 0} f(x)$?

The textbook also discusses one-sided limits and limits to infinity. You should be familiar with these. They will be discussed in class, also see the exercises at the end of this section.

2. The precise definition

The defintion from the text has two drawbacks.
Firstly, it is not precise enough for mathematics.
Secondly, it is wrong. Consider the following example :

Example 2.1. Define a function by

$$f(x) = \begin{cases} 1 & x \text{ is rational} \\ o & \text{otherwise} \end{cases}$$

What is $\lim_{x\to 0} f(x) =?$. It is both 0 and 1 according to the above definition.

Limits are vital for calculus and mathematics, and the current situation is not satisfactory.

Definition 2.1. Let $f(x)$ be a real valued function defined on some interval containing a. We say the the *limit of $f(x)$, as x approaches a, equals L* and write

$$\lim_{x\to a} f(x) = L$$

if for every $\epsilon > 0$ there is a $\delta > 0$ such that

$$0 < |x - a| < \delta \quad \text{implies} \quad |f(x) - L| < \epsilon.$$

This definition is quite a mouthful but is central to the development of mathematics. In this course you will need to be able to use this definition to compute simple limits, that is, limits of linear functions. We will not ask you to do anything more complicated than this.

Let us decode the definition a little. You should think of $\epsilon > 0$ as a small positive number, eg 0.00001. The requirement that

$$|f(x) - L| < \epsilon$$

means that $f(x)$ lies in the small interval

$$(L - \epsilon, L + \epsilon).$$

We need to find $\delta > 0$ so that this condition is true. Usually, δ will depend upon ϵ. So rephrasing, the last condition is that

$$x \in (a - \delta, a + \delta) \text{ and } x \neq a \implies f(x) \in (L - \epsilon, L + \epsilon).$$

Example 2.2. Use the $\epsilon - \delta$ definition of limit to show that

$$\lim_{x\to 0} 2x + 1 = 1.$$

So that in the definition of limit, we have $a = 0$, $L = 1$ and $f(x) = 2x + 1$.

In such problems you will need to do some rough work before you can present your solution. You should view this as a game against an imaginary opponent. Your opponent gives you an $\epsilon > 0$ and you need to produce a $\delta > 0$ so that

$$0 < |x| < \delta \quad \text{implies} \quad |(2x + 1) - 1| < \epsilon.$$

So we want :

$$\begin{aligned} \epsilon &> |2x + 1 - 1| \\ &= 2|x| \end{aligned}$$

Rearranging we find

$$|x| > \epsilon/2.$$

So we take $\delta = \epsilon/2$. This is the end of our rough work. The solution to the problem can now be presented below.

Solution : Given $\epsilon > 0$ we take $\delta = \epsilon/2$. Suppose

$$(\blacklozenge) \qquad 0 < |x| < \delta = \epsilon/2$$

we have

$$\begin{aligned} |f(x) - L| &= |2x + 1 - 1| \\ &= |2x| \\ &= 2|x| \\ &< 2(\epsilon/2) \quad \text{by } (\blacklozenge) \\ &= \epsilon. \end{aligned}$$

Example 2.3. Use the $\epsilon - \delta$ definition of limit to show that

$$\lim_{x \to 2} 3x + 1 = 7.$$

So that in the definition of limit, we have $a = 2$, $L = 7$ and $f(x) = 3x + 1$.
Rough : Given $\epsilon > 0$ we need to produce a $\delta > 0$ so that

$$0 < |x - 2| < \delta \quad \text{implies} \quad |(3x + 1) - 7| < \epsilon.$$

You can rewrite $f(x)$ as a function of $(x - 2)$ to make life a little easier. So

$$f(x) = 3x + 1 = 3(x - 2) + 7.$$

So we want :

$$\begin{aligned} \epsilon &> |f(x) - 7| \\ &= |3(x - 2) + 7 - 7| \\ &= 3|x - 2| \end{aligned}$$

Rearranging we find

$$|x - 2| > \epsilon/3.$$

So we take $\delta = \epsilon/3$. This is the end of our rough work. The solution to the problem can now be presented below.

Solution : Given $\epsilon > 0$ we take $\delta = \epsilon/3$. Suppose

$$(\heartsuit) \qquad 0 < |x - 2| < \delta = \epsilon/3$$

we have

$$\begin{aligned} |f(x) - L| &= |3(x - 2) + 7 - 7| \\ &= 3|x - 2| \\ &< 3(\epsilon/3) \quad \text{by } (\heartsuit) \\ &= \epsilon. \end{aligned}$$

3. Limsup and liminf

The function in example 2.1 has no limit as x goes to 0. However certain other important limits do exist. We define the lim sup and lim inf as follows :

$$\limsup_{x \to a} f(x) = \lim_{\epsilon \to 0} \sup\{f(x) | 0 < |x-a| < \epsilon\}$$

and

$$\liminf_{x \to a} f(x) = \lim_{\epsilon \to 0} \inf\{f(x) | 0 < |x-a| < \epsilon\}$$

Proposition 3.1. *(i)*

If $f(x)$ is bound below near a then $\limsup_{x \to a} f(x)$ and $\liminf_{x \to a} f(x)$ exist.

3.2. *$\lim_{x \to a} f(x)$ exists if and only if both $\limsup_{x \to a} f(x)$ and $\liminf_{x \to a} f(x)$ exist and are equal.*

Proof. Omitted for now. □

4. The limit laws

Please read the section on limit laws from the text. This will also be discussed in class. We summarize some of the important pieces of information below :

Theorem 4.1. *Suuppose that b is some number and*

$$\lim_{x \to a} f(x) \quad and \quad \lim_{x \to a} g(x) \quad exist.$$

It follows that :

4.1. $\lim_{x \to a}(f+g) = \lim_{x \to a} f + \lim_{x \to a} g$
4.2. $\lim_{x \to a}(bf) = b \lim_{x \to a} f$
4.3. $\lim_{x \to a}(f+g) = \lim_{x \to a} f - \lim_{x \to a} g$
4.4. $\lim_{x \to a}(f \cdot g) = \lim_{x \to a} f \cdot \lim_{x \to a} g$
4.5. *If $\lim_{x \to a} f(x) \neq 0$ then*

$$\lim_{x \to a} \frac{g}{f} = \frac{\lim_{x \to a} g}{\lim_{x \to a} f}.$$

4.6. *If $\lim_{x \to a} f(x)\, 0$ then*

$$\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$$

Proof. to be inserted here. □

Theorem 4.2. *If $f \leq g$ andd both limits at a exist then*

$$\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x)$$

Proof. to be inserted here. □

Theorem 4.3 (The Squeeze theorem)**.** *Suppose $f \leq g \leq h$ near a and all limits at a exist. If*

$$\lim_{x \to a} f(x) \leq \lim_{x \to a} h(x) = L$$

then

$$\lim_{x \to a} g(x) = L.$$

Proof. to be inserted here. □

5. Exercises

5.1. Please solve the following problems from section 2.2 of the text :
6th edition : 2, 5, 7, 12, 25, 26, 32
7th edition : 2, 5, 7, 11, 12, 29, 30, 36

5.2. Use the $\epsilon - \delta$ definition of limit to show that

$$\lim_{x\to 0} 5x + 9 = 9$$

5.3. Use the $\epsilon - \delta$ definition of limit to show that

$$\lim_{x\to 4} 5x + 9 = 29$$

5.4. Please solve the following problems from sectopm 2.3 of the text :
6th edition : 1, 2, 3-9, 10(a), odd 11-31, 33, 34, 35, 37, 39, 40, 47, 55, 57
7th edition : 1, 2, 3-9, 10(a), odd 11-29, 35, 36, 37, 39, 41, 42, 49, 57, 59

5.5. $\lim_{x\to 0^+} \frac{1}{\sqrt{x^2+2x^3}-x} =$

5.6. $\lim_{x\to \infty} \frac{1}{\sqrt{x^2+2x^3}-x} =$

5.7. $\lim_{x\to 0} x^3 \sin(1/x) =$ (reasoning is important)