LIMITS : PART 2

These lecture notes are designed to provide supplementary material to Stewart, "Single Variable Calculus, Seventh Edition, with Early Transcendentals". This is far from a complete, or even rigorous treatment of set theory. Such a treatment would lead us too far astray. This is just enough to get us going in mathematics.

As this is an enriched course, some of the material taught is for the interest of the student and will not appear on exams. This matherial is differentiated from examinable material by a blue font colour.

1. Limits at Infinity

We would like to give meaning to the symbol $\lim_{x\to\infty} f(x) = L$. In rough terms this means that $f(x)$ can be made arbitrarily close to L by making x sufficently large. The precise diefinition is :

Definition 1.1. We write $\lim_{x\to\infty} f(x) = L$ and say *that the limit as x goes to ∞ of $f(x)$ is L* if for every $\epsilon > 0$ there is an N such that $|x| > N$ implies

$$|f(x) - L| < \epsilon.$$

The symbol $\lim_{x\to-\infty} f(x) = L$ is defined similarly.

2. Continuity

Please read the section in the text on continuity. Remember that $f(x)$ is continuous at a means

$$\lim_{x\to a} f(x) = f(a).$$

Right continuous means

$$\lim_{x\to a^+} f(x) = f(a).$$

Left continuous means

$$\lim_{x\to a^-} f(x) = f(a).$$

An important property of continous functions is encapsulated by the intermediate value theorem :

Theorem 2.1. *Suppose that a function $f(x)$ is continuous on $[a, b]$. Suppose that α is a number between $f(a)$ and $f(b)$ then there is an $c \in [a, b]$ with $f(c) = \alpha$.*

There are two ways in which a function $f(x)$ may fail to be continuous at a. We could have that $lim_{x\to a}f(x)$ exist but

$$lim_{x\to a}f(x) \neq f(a)$$

or that

$$lim_{x\to a}f(x)$$

does not exist.

Example 2.1. The function defined by

$$f(x) = \begin{cases} x\sin(1/x) & x \neq 0 \\ 10 & x = 0 \end{cases}$$

is not continuous at 0. To see this note that by the squeeze theorem we have $\lim_{x\to 0} f(x) = 0$ while $f(0) = 10$.

Example 2.2. Consider the function

$$f(x) = \begin{cases} 1 & x \text{ rational} \\ 0 & \text{otherwise} \end{cases}$$

This function does not have a limit as x goes to 0, that is

$$lim_{x\to 0} f(x)$$

does not exist. Let us try to prove this using the $\epsilon - \delta$ definition.

The proof relies on the following property of the reals, in any interval (a, b) with $a < b$ there is at least one rational number and at least one non rational number in the interval.

We need to show that there is no number L with

$$lim_{x\to 0} f(x) = L.$$

Suppose that such an L existed. To derive a contradiction we need to negate the $\epsilon - \delta$ definition of limit. That is we need to find an ϵ for which there is no δ with the property that

$$0 < |x| < \delta \quad \text{implies} \quad |f(x) - L| < \epsilon.$$

Lets take $\epsilon = 1/3$, in fact anything smaller than or equal to $(1/2)$ will do. Suppose that there is a δ for this choice of ϵ, we derive a contradiction. By the above property of the reals there is a rational number say, x_1 with $0 < |x_1| < \delta$. So we must have

$$|f(x_1) - L| = |L - 1| < \epsilon = 1/3.$$

On the other hand, there is a non rational x_2 with $0 < |x_2| < \delta$. So we must have

$$|f(x_2) - L| = |L| < \epsilon = 1/3.$$

But this is impossible, as L cannot by smaller than $1/3$ and a distance of $1/3$ from 1. So no such δ exists and hence the limit does not exist as L was arbitrary.

3. Derivatives

It is important to know the definition of derivative :

$$\begin{aligned} f'(a) &= \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \\ &= \lim_{x\to a} \frac{f(x) - f(a)}{x - a}. \end{aligned}$$

The change of variables $x - a = h$ shows that these two definitions are the same.

Definition 3.1. We say that $f(x)$ is differentiable at a if the above limi exists.

Proposition 3.2. *If f is differentiable at a then it is continous at a.*

Proof. We have

$$\begin{aligned} \lim_{x \to a} f(x) - f(a) &= (\lim_{h \to 0} h)(\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}) \\ &= 0.f'(a). \end{aligned}$$

□

4. Questions

4.1. Please solve the following problems from section 2.5 of the text :
Seventh edition : 3 (a) (b), 12, 13, 17, 19, 21, 23, 35, 41, 45, 51, 53, 55

4.2. Please solve the follow problems from section 2.6 of the text :
Seventh edition : odd 15 - 37, 41, 45, 57(a)

4.3. Please solve the follow problems from section 2.7 of the text :
Seventh edition : 5, 7, 13, 31, 33, 35, 37, 53, 53.

4.4. Consider the following function

$$f(x) = \begin{cases} x^2 & x \text{ rational} \\ 0 & \text{otherwise} \end{cases}$$

Is $f(x)$ differentiable at $a = 0$? If so what is $f'(0)$?

4.5. Show carefully that the function $f(x) = |x|$ is not differentiable at $x = 0$.