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CALC 1501 LECTURE NOTES

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2. Factorization of Polynomials

2.1. **Complex Polynomials.** The set $\mathbb{R}$ of real numbers can be extended to a bigger set of the so-called *complex numbers*. This is done by introducing a single *imaginary* number $i = \sqrt{-1}$. Complex numbers can be written in the form $z = a + ib$, where $a, b \in \mathbb{R}$. In this representation a is called the *real* part of z, and b the *imaginary* part of z, denoted respectively by $\operatorname{Re} z$ and $\operatorname{Im} z$. Real numbers can be viewed as a subset of complex numbers with zero imaginary part. Thus, denoting the space of complex numbers by $\mathbb{C}$, we have the following chain of inclusions

$$\mathbb{N} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}.$$

We may extend the definition of arithmetic operations on real numbers to the space of complex numbers as follows:

(i) $(a + ib) + (a' + ib') = (a + a') + i(b + b')$
(ii) $(a + ib) \cdot (a' + ib') = (aa' - bb') + i(ab' + a'b)$
(iii) $\dfrac{a + ib}{a' + ib'} = \dfrac{aa' + bb'}{a'^2 + b'^2} + \dfrac{ba' - ab'}{a'^2 + b'^2} i$, if $a' + b'i \neq 0 = 0 + i0$.

One can verify that when $b = b' = 0$, the above formulas provide the usual operations of addition, multiplication and division for reals. Note that $i \cdot i = i^2 = -1$, which in particular means that the equation $z^2 + 1 = 0$ *over* the set of complex numbers has two complex roots: i and $-i$. This is in contrast with reals over which this equation has no solution.

With these operations on complex numbers we may define complex polynomials as functions on complex numbers defined by

$$P(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n, \quad \text{where } a_j \in \mathbb{C}. \tag{1}$$

If $a_0 \neq 0$, the n is called the degree of $P(z)$. In particular, if we ignore the choice of the letter for the unknown variable (x vs. z), the usual polynomials with real coefficients are examples of complex polynomials. (In other words, a polynomial is called *real* if in (1), $a_j \in \mathbb{R}$ for all j.) The following theorem is usually known as the *Fundamental Theorem of Algebra*.

Theorem 2.1. *Suppose $P(z)$ is a complex polynomial of degree $n > 0$. Then $P(z)$ has exactly n complex roots.*

In this theorem the number of roots should be counted with *multiplicity*, in other words, some roots may have to be counted more than once. For example, $z^2 + 2z + 1 = (z + 1)^2 = 0$ has two roots both of which are $z = -1$. In general, if $w_1, w_2, \ldots, w_m$ are the distinct roots of a polynomial $P(z)$, then we can write

$$P(z) = a_0 (z - w_1)^{k_1} (z - w_2)^{k_2} \ldots (z - w_m)^{k_m}. \tag{2}$$

This is called a *factorization* of a complex polynomial into complex linear factors. It is unique up to a change of order. The exponent k_j is called the *multiplicity* of the root w_j. The Fundamental Theorem of Algebra implies that $k_1 + k_2 + \cdots + k_m = n$. The proof of Theorem 2.1 requires some knowledge of complex analysis, a branch of mathematics that studies functions of complex variables.

Example 2.1. Consider the equation $f(z) = z^4 + z^2 = 0$. According to the Fundamental Theorem of Algebra, $f(z)$ has four roots. These can be easily found. Indeed, $z^4 + z^2 = z^2(z^2 + 1)$. Thus the roots are $z = 0$ (counted twice), $z = i$ and $z = -i$. So

$$f(z) = z^2(z - i)(z + i)$$

is a factorization of this polynomial into linear factors. $\diamond$

Note that not every real polynomial admits a factorization into *real* linear factors (e.g., $x^2 + 1$).

2.2. **Factorization of Real Polynomials.** An important operation on complex numbers is *complex conjugation*, or just conjugation, which is denoted by a horizontal bar, and defined as follows:

$$\overline{a + ib} = a - ib.$$

In other words, to conjugate a complex number we simply change the sign of the imaginary part of the number. Note that if z is a real number, then $\overline{z} = z$, i.e., conjugation leaves real numbers unchanged.

Let $w = a + ib$ be a complex number. Then $\overline{w} = a - ib$. Consider the expression$(z - w)(z - \overline{w})$. Then

$$(z - w)(z - \overline{w}) = z^2 - wz - \overline{w}z + w\overline{w} = z^2 - (w + \overline{w})z + w\overline{w}. \tag{3}$$

We have $w + \overline{w} = (a + ib) + (a - ib) = 2a$, and $w\overline{w} = (a + ib)(a - ib) = a^2 + b^2$. Both are real numbers. Thus the product of two monomials as above with conjugate free terms yields a degree two polynomial with real coefficients.

Suppose now

$$P(z) = z^n + b_1 z^{n-1} + \cdots + b_{n-1} z + b_n, \quad \text{where} \quad b_j \in \mathbb{R}, \tag{4}$$

is a polynomial of degree n with real coefficients, and let ζ be a complex root of $P(z)$. Then

$$\zeta^n + b_1 \zeta^{n-1} + \cdots + b_{n-1}\zeta + b_n = 0.$$

Conjugation of both sides of this equation gives

$$\overline{\zeta}^n + b_1 \overline{\zeta}^{n-1} + \cdots + b_{n-1}\overline{\zeta} + b_n = 0.$$

Note that coefficients b_j did not change because conjugation does not change real numbers. We also used here the fact that for $z, w \in \mathbb{C}$, we have $\overline{z + w} = \overline{z} + \overline{w}$, and $\overline{z \cdot w} = \overline{z} \cdot \overline{w}$, which can be verified directly. What the last equation tells us is that $\overline{\zeta}$ is also a root of $P(z)$. In other words, if ζ is a complex root of $P(z)$ and ζ is not a real number, then $\overline{\zeta}$ is also a root of $P(z)$. Thus we may write

$$P(z) = (z - x_1)(z - x_2) \cdots (z - x_m)(z - z_1)(z - \overline{z_1}) \cdots (z - z_k)(z - \overline{z_k}), \tag{5}$$

where $x_1, \ldots, x_m$ are the real roots of $P(x)$, and $z_1, \overline{z_1}, \ldots, z_k, \overline{z_k}$ are the pairs of complex roots and their conjugates. Using the calculation in (3) we have

$$(z - z_1)(z - \overline{z_1}) = z - A_1 z + B_1, \quad \text{where} \quad A_1 = 2\mathrm{Re}\, z_1, \text{ and } B_1 = (\mathrm{Re}\, z_1)^2 + (\mathrm{Im}\, z_1)^2,$$

and similarly for the other pairs of complex conjugate roots of $P(z)$. Using this, and replacing z with x in (5) yields

$$P(x) = (x - x_1)(x - x_2) \cdots (x - x_m)(x^2 - A_1 x + B_1)(x^2 - A_2 x + B_2) \cdots (x^2 - A_k x + B_k),$$

where all the coefficients are real numbers. Thus, we proved the following theorem.

Theorem 2.2. *Suppose $P(x)$ is a real polynomial of degree $n > 0$. Then $P(x)$ admits factorization into a product of linear and quadratic factors with real coefficients.*

This theorem is used in the theory of integration of rational functions using partial fractions.

Example 2.2. Let $P(x) = x^4 + 1$. This polynomial does not have any real roots. Nevertheless, according to Theorem 2.2, it can be factored into a product of two real polynomials. But what are these? One possible solution would be to find complex roots of $P(z)$ and then to multiply the conjugate monomials as discussed above. However, finding complex roots is not an easy task. Instead, we can try to factorize $P(z)$ into two polynomials $x^2 + ax + 1$ and $x^2 + bx + 1$ for some $a, b \in \mathbb{R}$. We get

$$(x^2 + ax + 1)(x^2 + bx + 1) = x^4 + ax^3 + x^2 + bx^3 + abx^2 + bx + x^2 + ax + 1.$$

We set this equal to $x^4 + 1$ and compare the coefficients of x^3, x^2, and x. It follows that $a = -b$ and $ab = -2$. So we may take $a = \sqrt{2}$ and $b = -\sqrt{2}$. This gives the required factorization:

$$x^4 + 1 = (x^2 + \sqrt{2}\,x + 1)(x^2 - \sqrt{2}\,x + 1).$$

$\diamond$