v: 2013-01-21

CALC 1501 LECTURE NOTES

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3. Improper Intergrals.

So far we dealt with integration of continuous functions on bounded intervals. In this section we will discuss integration of continuous functions on unbounded intervals, and also integration of certain unbounded functions.

3.1. **Unbounded intervals.** Suppose a function $f(x)$ is continuous on the interval (a, ∞), so that the integral $\int_a^b f(x)dx$ is well-defined for any $b > a$. The limit of this integral as $b \to \infty$ will be called the *improper integral of* $f(x)$ *on* (a, ∞). That is

$$\int_a^\infty f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx. \tag{3.1}$$

If the limit exists (i.e., it is a finite number), then we say that the integral $\int_a^\infty f(x)dx$ *converges* or *is convergent*. If the limit is infinite, or does not exist, we say that the improper integral *diverges* or *is divergent*.

Example 3.1. Consider $f(x) = \frac{1}{1+x^2}$. Then for any $b > 0$,

$$\int_0^b \frac{dx}{1+x^2} = \tan^{-1} x\Big|_0^b = \tan^{-1} b.$$

Therefore,

$$\int_0^\infty \frac{dx}{1+x^2} = \lim_{b\to\infty} \int_0^b \frac{dx}{1+x^2} = \lim_{b\to\infty} \tan^{-1} b = \frac{\pi}{2}.$$

Thus, the integral $\int_0^\infty \frac{dx}{1+x^2}$ converges to $\pi/2$. $\diamond$

Example 3.2. Consider $f(x) = \frac{1}{x^p}$, where $p > 0$ is a real number. Let us find the values of the exponent p for which the integral

$$\int_1^\infty \frac{dx}{x^p} \tag{3.2}$$

converges. If $p \neq 1$, and $b > 1$ any number, then

$$\int_1^b \frac{dx}{x^p} = \frac{1}{1-p}\, x^{1-p}\Big|_1^b = \frac{1}{1-p}(b^{1-p} - 1).$$

If $p > 1$, then $b^{1-p} \to 0$ as $a \to \infty$, and the integral in (3.2) converges. If $p < 1$, then $b^{1-p} \to \infty$ and the integral diverges. Suppose now that $p = 1$. Then

$$\int_1^b \frac{dx}{x} = \ln x\Big|_1^b = \ln b,$$

and since $\lim\limits_{b\to\infty} \ln b = \infty$, we conclude that $\int_1^\infty \frac{dx}{x}$ diverges. Thus, the improper integral in (3.2) converges for $p > 1$ and diverges for $0 < p \leq 1$. $\diamond$

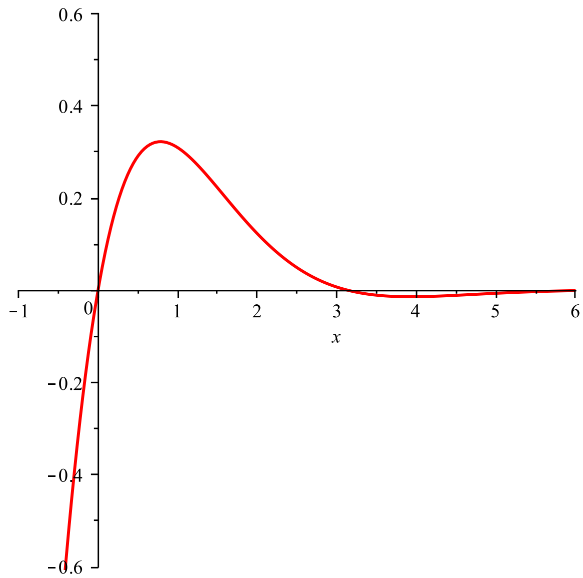

Figure 1. The graph of $e^{-x}\sin x$

In a similar way we define improper integrals from $-\infty$ to b:

$$\int_{-\infty}^{b} f(x)dx = \lim_{a\to-\infty}\int_{a}^{b} f(x)dx.$$

Finally, we define

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{\substack{b\to\infty \\ a\to-\infty}}\int_{a}^{b} f(x)dx.$$

In the latter case, we may also choose any number A so that

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{A} f(x)dx + \int_{A}^{\infty} f(x)dx.$$

Then the integral on the left-hand side converges if and only if both integrals on the right-hand side converge.

Example 3.3. Determine whether the following integrals converge.

(i) $\displaystyle\int_{-\infty}^{0} e^x \sin x\, dx$

(ii) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2-2x+2}$

Solution: (i) Integrating by parts twice, we obtain

$$\int e^x \sin x\, dx = \frac{1}{2}(-e^x\cos x + e^x \sin x).$$

Therefore,

$$\int_{-\infty}^{0} e^x \sin x\, dx = \lim_{a-\to\infty}\frac{1}{2}(-e^x\cos x + e^x\sin x)\Big|_a^0 = \lim_{a\to-\infty}\frac{1}{2}(-1+e^a\cos a - e^a \sin a) = -\frac{1}{2}.$$

Indeed, since $\sin x$ and $\cos x$ are bounded functions, and $\lim_{b\to\infty} e^{-b} = 0$, it follows that

$$\lim_{b\to\infty} e^{-b}\sin b = \lim_{b\to\infty} e^{-b}\cos b = 0.$$

(ii) First note that denominator of the integrand does not vanish for any x, and so the function under the integral sign is continuous on $\mathbb{R}$. We have

$$\int_{-\infty}^{\infty} \frac{dx}{x^2-2x+2} = \int_{-\infty}^{1} \frac{dx}{x^2-2x+2} + \int_{1}^{\infty} \frac{dx}{x^2-2x+2}.$$

Here instead of 1 as a limit of integration we could have chosen any other number. To determine convergence of each integral on the right, observe that

$$\int \frac{dx}{x^2-2x+2} = \int \frac{dx}{(x-1)^2+1} = \tan^{-1}(x-1).$$

Thus,

$$\int_{-\infty}^{1} \frac{dx}{x^2-2x+2} = \lim_{a\to-\infty} \tan^{-1}(x-1)\Big|_a^1 = \lim_{a\to-\infty} \big(\tan^{-1}(0) - \tan^{-1}(a-1)\big) = \pi/2.$$

Similarly,

$$\int_{1}^{\infty} \frac{dx}{x^2-2x+2} = \lim_{b\to\infty} \tan^{-1}(x-1)\Big|_1^b = \lim_{b\to\infty} \big(\tan^{-1}(b-1) - \tan^{-1}(0)\big) = \pi/2,$$

and so

$$\int_{-\infty}^{\infty} \frac{dx}{x^2-2x+2} = \pi.$$

$\diamond$

Sometimes it is possible to determine if an improper integral converges without computing the limit. This is particularly useful when the integral is hard or simply impossible to evaluate explicitly. For example, we can use the following result, usually referred to as the *Comparison test for improper integrals.*

Theorem 3.1. *Suppose there exists a number $A \ge a$ such that the inequality*

$$0 \le f(x) \le g(x)$$

holds for all $x \ge A$. Then convergence of the integral $\int_a^\infty g(x)dx$ implies convergence of $\int_a^\infty f(x)dx$, and equivalently, divergence of $\int_a^\infty f(x)dx$ implies divergence of $\int_a^\infty g(x)dx$.

We will return to the proof of this theorem later, when we discuss series. Note that the fact that the required inequality must hold starting only from some number A, which can be quite large, indicates that for the convergence of the integral only the behaviour of the function at infinity matters.

Example 3.4. We determine convergence of

$$\int_1^\infty \frac{\sin^2 x}{\sqrt{x^3+1}}dx. \tag{3.3}$$

Since $\sin^2 x \le 1$, and $\sqrt{x^3+1} > \sqrt{x^3}$ for $x > 1$, we conclude that

$$\frac{\sin^2 x}{\sqrt{x^3+1}} \le \frac{1}{\sqrt{x^3}} = \frac{1}{x^{3/2}}.$$

By Example 3.2, the integral $\int_1^\infty \frac{dx}{x^p}$ converges for $p = 3/2$. Therefore, by Theorem 3.1, the improper integral in (3.3) converges. ⋄

Example 3.5. Consider

$$\int_2^\infty \frac{\ln(\ln x)}{\sqrt{x}} dx.$$

For $x > e^3 \approx 20.085537$, we have $\ln(\ln x) > 1$, and therefore, $\frac{\ln(\ln x)}{\sqrt{x}} > \frac{1}{\sqrt{x}}$. Since the integral $\int_1^\infty \frac{dx}{\sqrt{x}}$ diverges by Example 3.2, the integral above diverges by Theorem 3.1. ⋄

3.2. **Unbounded functions.** Suppose now that the interval of integration is bounded but the function is unbounded. We first assume that the function $f(x)$ is defined and continuous on $[a, b)$ but becomes unbounded as $x \to b^-$. For such function $f(x)$ the integral $\int_a^b f(x)dx$ cannot be defined using the Riemann sums as we did this for continuous functions on $[a, b]$. Instead, we set

$$\int_a^b f(x)dx = \lim_{t \to b^-} \int_a^t f(x)dx. \tag{3.4}$$

We say that the improper integral in (3.4) *converges* if the limit exists and finite, and *diverges* otherwise.

Example 3.6. The function $f(x) = \frac{1}{\sqrt{1-x^2}}$ is continuous on $[0, 1)$, and for any $0 < b < 1$, we have

$$\int_0^b \frac{dx}{\sqrt{1 - x^2}} = \sin^{-1} x \Big|_0^b = \sin^{-1}(b).$$

Thus,

$$\int_0^1 \frac{dx}{\sqrt{1 - x^2}} dx = \lim_{b \to 1^-} \sin^{-1}(b) = \pi/2,$$

and the integral converges. ⋄

A similar definition can be made when the function $f(x)$ is unbounded near the point a:

$$\int_a^b f(x)dx = \lim_{t \to a^+} \int_t^b f(x)dx. \tag{3.5}$$

Example 3.7. Verify convergence of the improper integral

$$\int_{-1}^1 \frac{\cos^{-1}(x)}{\sqrt{1 - x^2}} dx.$$

First observe that when $x \to -1^+$, the function $\cos^{-1}(x)$ approaches π and so

$$\lim_{x \to -1^+} \frac{\cos^{-1}(x)}{\sqrt{1 - x^2}} = \infty.$$

On the other hand, using L'Hôpital's Rule,

$$\lim_{x \to 1^-} \frac{\cos^{-1}(x)}{\sqrt{1 - x^2}} = 1,$$

and therefore, the integral is proper near $x = 1$. Hence,

$$\int_{-1}^1 \frac{\cos^{-1}(x)}{\sqrt{1 - x^2}} dx = \lim_{t \to -1^+} \int_t^1 \frac{\cos^{-1}(x)}{\sqrt{1 - x^2}} dx.$$

We have

$$\int_t^1 \frac{\cos^{-1}(x)}{\sqrt{1-x^2}}\,dx = -\int_{\cos^{-1}(t)}^0 u du = -\frac{1}{2}u^2\Big|_{\cos^{-1}(t)}^0 = -\frac{1}{2}(\cos^{-1}x)^2\Big|_t^1 = \frac{1}{2}(\cos^{-1}t)^2.$$

where $u=\cos^{-1}(x)$. Thus,

$$\int_{-1}^1 \frac{\cos^{-1}(x)}{\sqrt{1-x^2}}\,dx = \lim_{t\to -1^+}\frac{1}{2}(\cos^{-1}t)^2 = \frac{\pi^2}{2}.$$

◇

Finally, suppose that there is a point $c\in(a,b)$ such that the function $f(x)$ is continuous on $[a,b]$ except the point c near which it is unbounded. Then the improper integral of $f(x)$ on (a,b) is defined as follows:

$$\int_a^b f(x)dx = \lim_{t\to c^-}\int_a^t f(x)dx + \lim_{s\to c^+}\int_s^b f(x)dx.$$

Note that the integral on the left converges if and only if both integral on the right converge. Also, it would be incorrect to integrate the function $f(x)$ on (a,b) ignoring the singularity of $f(x)$ at c.

Example 3.8. Consider

$$\int_0^2 \frac{dx}{x^2-1}.$$

The function under the integral sign is discontinuous at the point $x=1$, therefore the above integral is improper, and

$$\int_0^2 \frac{dx}{x^2-1} = \lim_{t\to 1^-}\int_0^t \frac{dx}{x^2-1} + \lim_{s\to 1^+}\int_s^2 \frac{dx}{x^2-1}.$$

Consider the first integral on the right:

$$\int_0^t \frac{dx}{x^2-1} = \frac{1}{2}\ln\frac{|x-1|}{|x+1|}\Bigg|_0^t = \frac{1}{2}\ln\frac{|t-1|}{|t+1|}.$$

As $t\to 1^-$, the quantity under the logarithm approaches zero, and therefore, this integral diverges as $t\to 1^-$. Therefore, $\int_0^2 \frac{dx}{x^2-1}$ also diverges. ◇

3.3. **The Gamma function.** Recall that the factorial function $n!$ is defined on the set of natural numbers as $n! = 1\cdot 2\cdot \ldots \cdot n$. This definition is not suitable for real numbers. However, we may plot the points $(n,n!)$ on the (x,y)-coordinate system, and raise the following important interpolation problem: find a smooth curve that connects the points (x,y) in the plane given by $(n,n!)$, $n\in\mathbb{N}$. If such a function indeed exists, then we may simply declare the value of this function on non-integers to be the value of the factorial.

A plot of the first few factorials (see Fig 1.) makes it clear that such a curve can be drawn, but how can one derive a formula that precisely describes the curve? As it turns out, no finite combination of power functions, exponential functions or logarithms with a fixed number of terms can produce such a function. But it is possible to find a general formula as an integral depending on a parameter. This is called *the Gamma function*, $\Gamma(x)$. It was discovered by L. Euler in 1729. The symbol $\Gamma(x)$ and the name were proposed in 1814 by A.M. Legendre.

The Gamma function $\Gamma(x)$ is defined as an improper integral

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt. \tag{3.6}$$

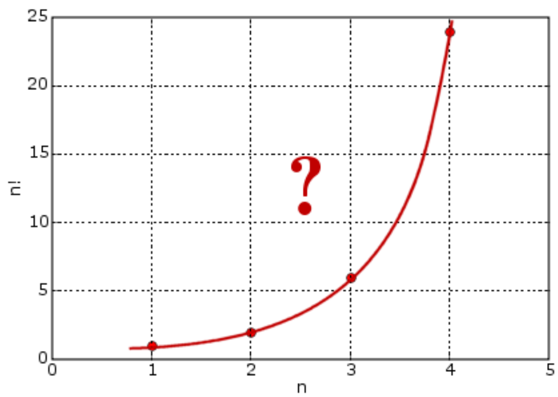

FIGURE 2. Interpolating $n!$

This brings together integration by parts and improper integrals. First consider the case $x = 1$. We have

$$\Gamma(1) = \int_0^\infty e^{-t}dt = \lim_{s\to\infty}\int_0^s e^{-t}dt = \lim_{s\to\infty} -e^{-t}\Big|_0^s = 1.$$

Now, using integration by parts, we can show that $\Gamma(n+1) = n\cdot\Gamma(n)$. Indeed, for an integer $n \geq 1$,

$$\Gamma(n+1) = \int_0^\infty t^{n+1-1}e^{-t}dt = \int_0^\infty t^n e^{-t}dt.$$

Consider the indefinite integral $\int t^n e^{-t}dt$. We apply integration by parts by choosing $u = t^n$, and $dv = e^{-t}dt$. Then $du = n\,t^{n-1}dt$ and $v = -e^{-t}$. According to the integration by parts formula, we have

$$\int t^n e^{-t}dt = -t^n\,e^{-t} - \int -e^{-t}n\,t^{n-1}dt = -t^n\,e^{-t} + n\int t^{n-1}e^{-t}dt.$$

Thus,

$$\int_0^\infty t^n e^{-t}dt = \lim_{s\to\infty}\left[-t^n\,e^{-t}\Big|_0^s + n\int_0^s t^{n-1}e^{-t}dt\right]. \tag{3.7}$$

For the first term inside the limit above we get

$$\lim_{s\to\infty}\left(\frac{-t^n}{e^t}\right)\Big|_0^s = \lim_{s\to\infty} -\frac{s^n}{e^s}.$$

Using L'Hôpital's Rule n times we see that

$$\lim_{s\to\infty}\frac{s^n}{e^s} = \lim_{s\to\infty}\frac{n!\,s^0}{e^s} = 0.$$

For the second term on the right hand side of (3.7) we have

$$\lim_{s\to\infty} n\int_0^s t^{n-1}e^{-t}dt = n\Gamma(n).$$

Combining everything together we have $\Gamma(n+1) = n\Gamma(n)$. This identity provides a reduction formula which can be used to compute inductively the values of the Gamma function for positive integers:

$$\Gamma(n+1) = n! \quad \text{where} \quad n \in \mathbb{N}.$$

Indeed, $\Gamma(2) = 1$; $\Gamma(3) = \Gamma(2+1) = 2\cdot\Gamma(2) = 2$; $\Gamma(4) = \Gamma(3+1) = 3\cdot\Gamma(3) = 3\cdot 2$, etc.

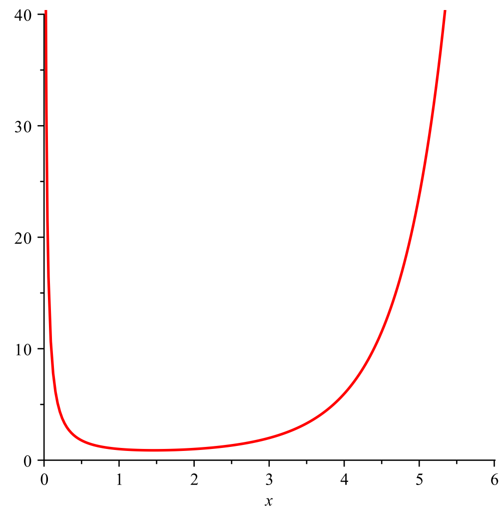

FIGURE 3. The graph of $\Gamma(x)$

In fact, by inspection we see that our application of the integration by parts formula is valid not only for integer values n, but for all real $x > 0$ (see Exercises 3.4 and 3.5 for the case $0 < x < 1$), and so we have

$$\Gamma(x+1) = x\,\Gamma(x) \quad \text{for all} \quad x > 0. \tag{3.8}$$

3.4. **Exercises.**

3·1. Give an $\epsilon - \delta$ definition of convergence of an improper integral in equation (3.4).

3·2. Determine the convergence of the improper integral. If converges, find its value.

(a) $\displaystyle\int_0^\infty e^{-ax} \ \sin bx\, dx$

(b) $\displaystyle\int_{2/\pi}^\infty \frac{1}{x^2} \sin\frac{1}{x} \ dx$

(c) $\displaystyle\int_0^\infty \cos x\, dx$

3·3. Determine the convergence of the integral, but do not evaluate.

(a) $\displaystyle\int_0^\infty \frac{\sin x}{1+x^2} dx$

(b) $\displaystyle\int_1^\infty \frac{\tan^{-1} x}{\cos x + x^2} dx$

3·4. Abel's test for convergence of improper integrals states the following: suppose $f(x)$ and $g(x)$ are defined on $[a, \infty)$, $\int_a^\infty f(x)dx$ converges, and there exists a real number M such that $|g(x)| < M$ for all $x \geq a$. Then the integral

$$\int_a^\infty f(x)g(x)dx$$

converges. Use this test to verify convergence of the following integrals:

(a) $\displaystyle\int_1^\infty \frac{\sin x}{\sqrt{x^4+1}} \ dx,$

(b) $\displaystyle\int_1^\infty \frac{\sin x}{\sqrt{x^4+1}} \cdot \tan^{-1} x \, dx.$

3·5. Determine for which values of $\lambda \in \mathbb{R}$ the following integral converges:

(a) $\displaystyle\int_0^1 x^\lambda \ln x \, dx,$

(b) $\displaystyle\int_0^1 \frac{dx}{x^\lambda}.$

(c) $\displaystyle\int_0^\infty \frac{dx}{x^\lambda}.$

3·6. Determine the convergence of the improper integral. If converges, find its value.

(a) $\displaystyle\int_{-1}^1 \frac{dx}{\sqrt{1-x^2}}$

(b) $\displaystyle\int_0^1 \ln x \, dx$

(c) $\displaystyle\int_1^2 \frac{1}{x \ln x} \, dx$

(d) $\displaystyle\int_0^1 \frac{\ln x}{\sqrt{x}} \, dx$

3·7. For $x \geq 1$ our calculations in Section 3.3 show the convergences of the improper integral that defines the Gamma function. However, if $x < 1$, then the integral in (3.6) contains a negative power of t ($x - 1$ becomes negative). Use the comparison test for improper integrals to show that the Gamma function is well-defined for $0 < x < 1$. (*Hint:* split the integral in (3.6) into two integrals.)

3·8. Verify formula (3.8) for the case when $0 < x < 1$.

3·9. Show that the integral in (3.6) diverges if $x \leq 0$.

3·10. The integral

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

is called the *Gaussian* integral. It is particularly important in probability theory and statistics.

(a) Use the Comparison test to prove that the Gaussian integral converges (without referring to its actual value, which is not so easy to compute).

(b) Use the value of this integral to evaluate $\Gamma(1/2)$

3·11. Use Problem 3.7 to calculate $\Gamma(5/2)$.

3·12. Prove that $\lim_{x\to 0^+} \Gamma(x) = +\infty$.