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CALC 1501 LECTURE NOTES

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2. Techniques of Integration

The rules of differentiation give us an explicit algorithm for calculating derivatives of all elementary functions, including trigonometric and exponential functions, as well as logarithms. By comparison, integration of elementary functions in general is a more difficult task. In fact, some integrals, such as

$$\int e^{-x^2}\,dx, \int \sin(x^2)\,dx, \int \frac{\sin x}{x}dx, \int \frac{dx}{\ln x}$$

cannot be expressed as elementary functions. To understand better this striking phenomenon, recall that $\int \frac{dx}{x} = \ln x$, and we see that integration of a rational function leads to a *transcendental* function. So one may expect that integration of transcendental functions leads to an even bigger class of functions that cannot be expressed as combinations of elementary functions, although from the general theory of integration we know that these functions exist and are well-defined.

In this lecture we discuss some more advanced techniques of integration.

2.1. **Integration by Parts.** Let $u = f(x)$ and $v = g(x)$ be differentiable functions with continuous derivatives. Then by the product rule: $d(uv) = udv + vdu$, or

$$udv = d(uv) - vdu. \tag{2.1}$$

The antiderivative of the expression $d\,(uv)$ is uv, and therefore, by integrating both sides of (2.1) we obtain the formula of *integration by parts*:

$$\int u\,dv = uv - \int v\,du. \tag{2.2}$$

The integration by parts formula can be used, for example, for integration of products of functions.

Example 2.1. To evaluate $\int x\cos x dx$ we let $u = x$, $dv = \cos x dx$, so that $du = dx$ and $v = \sin x$, and use (2.2):

$$\int x\cos x dx = x\sin x - \int sinx dx = x\sin x + \cos x + C. \tag{2.3}$$

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Application of the integration by parts formula requires breaking the integrand into two parts: u and dv, the first of which should be differentiated and the second integrated. The choice of u and dv should be such that integration of dv is relatively simple and that the the resulting integral is simpler than the original.

Example 2.2. To compute $\int x^3 \ln x\,dx$ we set $u = \ln x$, $dv = x^3 dx$. Then

$$\int x^3 \ln x\,dx = \frac{1}{4}\,x^4 \ln x - \frac{1}{4}\int x^3 dx = \frac{1}{4}\,x^4 \ln x - \frac{1}{16}\,x^4 + C.$$

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If the integrand is not a product of functions, then the choice $dx = dv$ may lead to a simplification of the integral.

Example 2.3.
$$\int \ln x\,dx = x\ln x - \int dx = x(\ln x - 1) + C.$$
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Sometimes it is necessary to apply integration by parts several times.

Example 2.4.
$$\begin{aligned}\int x^2 \sin x\,dx &= \left| u = x^3;\ dv = \sin x dx \right| \\ &= -x^2\cos x + 2\int x\cos x\,dx = \left| u = x;\ dv = \cos x\,dx \right| \\ &= -x^2\cos x + 2(x\sin x + \cos x) + C.\end{aligned}$$
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Integration by parts can be used to evaluate definite integrals. In this case the formula of integration by parts becomes

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du. \tag{2.4}$$

Example 2.5.
$$\begin{aligned}\int_0^1 \tan^{-1} x\,dx &= \left| u = \tan^{-1} x;\ dv = dx \right| = x\tan^{-1} x\Big|_0^1 - \int_0^1 \frac{x\,dx}{1+x^2} \\ &= \tan^{-1}(1) - \frac{1}{2}\ln(x^2+1)\Big|_0^1 = \frac{\pi}{4} - \frac{1}{2}\ln 2.\end{aligned}$$
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Sometimes integration by parts can be used to obtain an equation that can be solved for the unknown integral.

Example 2.6. To evaluate $\int \cos x\, e^x dx$, we let $u = \cos x$, $dv = e^x\,dx$ and use (2.2):
$$\int \cos x\, e^x\,dx = \cos x\, e^x + \int \sin x\, e^x\,dx.$$
We did not seem to achieve any visible simplification, so we apply integration by parts again to the last integral above:
$$\int \sin x\, e^x\,dx = \sin x\, e^x - \int \cos x\, e^x\,dx.$$
Combining everything together we obtain:
$$\int \cos x\, e^x\,dx = \cos x\, e^x + \left(\sin x\, e^x - \int \cos x\, e^x\,dx\right).$$
By solving for $\int \cos x\, e^x\,dx$, we obtain
$$\int \cos x\, e^x\,dx = \frac{1}{2}(\cos x + \sin x)e^x + C.$$
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Successive use of integration by parts often leads to recursive (or reduction) formulas.

Example 2.7. Let

$$J_n = \int \frac{dx}{(x^2+1)^n}. \tag{2.5}$$

Prove the recurrent formula

$$J_{n+1} = \frac{1}{2n}\left(\frac{x}{(x^2+1)^n} + (2n-1)J_n\right). \tag{2.6}$$

Solution: we use integration by parts for J_n. Let

$$u = \frac{1}{(x^2+1)^n}, \quad dv = dx.$$

Then

$$du = \frac{-2nx\,dx}{(x^2+1)^{n+1}}, \quad v = x.$$

Therefore,

$$\begin{aligned} J_n &= \frac{x}{(x^2+1)^n} + 2n\int \frac{x^2\,dx}{(x^2+1)^{n+1}} = \frac{x}{(x^2+1)^n} + 2n\int \frac{(x^2+1)-1}{(x^2+1)^{n+1}}dx \\ &= \frac{x}{(x^2+1)^n} + 2nJ_n - 2nJ_{n+1}. \end{aligned}$$

Formula (2.6) now can be obtained by solving the above identity for J_{n+1}.

2.2. **Integration by Partial Fractions.** Recall that a *rational* function is the quotient of two polynomials. In this section we describe a general algorithm for integration of rational functions.

2.2.1. *Basic Fractions.* First we consider four simple rational functions, which we call the *basic fractions*:

$$I.\ \frac{A}{x-a}, \quad II.\ \frac{A}{(x-a)^k}, \quad III.\ \frac{Bx+C}{x^2+bx+c}, \quad IV.\ \frac{Bx+C}{(x^2+bx+c)^k}, \tag{2.7}$$

where k is an integer bigger than one. We assume that in III and IV the quadratic terms are irreducible, i.e., they cannot be factorized into linear factors. This is equivalent to requiring that $b^2-4c<0$. Integration of the first two basic fractions is simple:

$$\int \frac{A}{x-a}dx = A\ln|x-a| + C,$$
$$\int \frac{A}{(x-a)^k}dx = \frac{A}{1-k}\frac{1}{(x-a)^{1-k}} + C.$$

For fractions III and IV we can complete the square

$$x^2+bx+c = \left(x+\frac{b}{2}\right)^2 + \left(c-\frac{b^2}{4}\right),$$

and use the change of variables $t = x + \frac{b}{2}$. This will simplify the integral.

Example 2.8.

$$\int \frac{xdx}{(x^2+2x+2)^2} = \int \frac{xdx}{((x+1)^2+1)^2} = \Big| t = x+1, dt = dx \Big|$$
$$= \int \frac{(t-1)\,dt}{(t^2+1)^2} = \int \frac{tdt}{(t^2+1)^2} - \int \frac{dt}{(t^2+1)^2}.$$

For the first integral we use the substitution $y = t^2 + 1$ to get

$$\int \frac{tdt}{(t^2+1)^2} = \frac{1}{2}\int \frac{dy}{y^2} = -\frac{1}{2(t^2+1)} = -\frac{1}{2(x^2+2x+2)}.$$

For the second integral we can use the reduction formula (2.6):

$$\int \frac{dt}{(t^2+1)^2} = J_2 = \frac{1}{2}\left(\frac{t}{t^2+1} + J_1\right) = \frac{1}{2}\left(\frac{t}{t^2+1} + \tan^{-1} t\right).$$

Thus, we obtain

$$\int \frac{xdx}{(x^2+2x+2)^2} = -\frac{1}{2(x^2+2x+2)} - \frac{1}{2}\left(\frac{x+1}{x^2+2x+2} + \tan^{-1}(x+1) + C\right) =$$
$$= -\frac{x+2}{2(x^2+2x+2)} - \frac{1}{2}\tan^{-1}(x+1) + C.$$

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2.2.2. *Reduction to basic fractions.* We call a rational function $\frac{P(x)}{Q(x)}$ *proper*, if $\deg P < \deg Q$. Using long division of polynomials, any rational function can be represented as the sum of a polynomial and a proper rational function.

Example 2.9. The rational function $\frac{3x^3-5x^2+10x-3}{3x+1}$ is not proper. Its proper representation has the form

$$\frac{3x^3-5x^2+10x-3}{3x+1} = x^2 - 2x + 4 - \frac{7}{3x+1}.$$

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An important general fact from abstract algebra is the following theorem

Theorem 2.1. *(i) Any polynomial of degree at least one admits factorization into a product of linear and irreducible quadratic factors.*

(ii) Any proper rational function can be represented as a finite sum of basic fractions.

The proof of part (i) of the theorem requires the knowledge of *complex* numbers. You can find more details about this in the Appendix (this material is not part of the course curriculum). The proof of part (ii) relies on part (i), but does not require additional knowledge beyond what we already know.

As a corollary to Theorem 2.1 we conclude that the integral of any rational function can be given by elementary functions. For simplicity we only present the final algorithm for the *partial fraction decomposition* that allows us to integrate any rational function. Let $\frac{P(x)}{Q(x)}$ be a proper rational function. According to Theorem 2.1(i) we may factor $Q(x)$ into linear and quadratic factors:

$$Q(x) = C \cdot (x-a_1)^{m_1} \cdot \cdots \cdot (x-a_k)^{m_k} \cdot (x^2+b_1x+c_1)^{n_1} \cdot \cdots \cdot (x^2+b_lx+c_l)^{n_l}, \tag{2.8}$$

where $x^2+b_jx+c_j$ are irreducible quadratic terms and m_j, n_j are positive integers. We will consider several cases depending on the factorization above.

Case 1. Factorization (2.8) contains only unrepeated linear terms. In this case the following partial fraction decomposition always holds.

$$\frac{P(x)}{Q(x)} = \frac{P(x)}{(x-a_1)\cdot\dots\cdot(x-a_k)} = \frac{A_1}{x-a_1} + \dots + \frac{A_k}{x-a_k}, \tag{2.9}$$

where A_j are some coefficients that can be found by forming a suitable system of linear equations.

Example 2.10. Evaluate

$$\int \frac{x\,dx}{(x+1)(x+2)(x-3)}.$$

Solution: we write

$$\frac{x}{(x+1)(x+2)(x-3)} = \frac{A_1}{x+1} + \frac{A_2}{x+2} + \frac{A_3}{x-3}.$$

After bringing the right-hand side to a common denominator and equating the numerators on the left and on the right we obtain

$$x = A_1(x+2)(x-3) + A_2(x+1)(x-3) + A_3(x+1)(x+2).$$

By letting $x = -1, -2$, and 3 we find that

$$-1 = -4A_1, \quad -2 = 5A_2, \quad 3 = 20A_3,$$

i.e., $A_1 = \frac{1}{4}$, $A_2 = -\frac{2}{5}$, $A_3 = \frac{3}{20}$. Therefore,

$$\int \frac{x\,dx}{(x+1)(x+2)(x-3)} = \frac{1}{4}\ln|x+1| - \frac{2}{5}\ln|x+2| + \frac{3}{20}\ln|x-3| + C.$$

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Remark: in the above example one can choose arbitrary values of x to find the coefficients A_j. In general one will obtain a system of linear equations in A_j that can be solved using elimination of variables or Cramer's method. However, the choice $x = -1, -2$, and 3 makes the computations particularly simple.

Case 2. Factorization (2.8) contains only linear terms some of which appear more than once. In this case the partial fraction decomposition becomes

$$\begin{aligned}\frac{P(x)}{Q(x)} &= \frac{P(x)}{(x-a_1)^{m_1}\cdot\dots\cdot(x-a_k)^{m_k}} = \\ &= \frac{A_{11}}{x-a_1} + \frac{A_{12}}{(x-a_1)^2} + \dots + \frac{A_{1k}}{(x-a_1)^{m_k}} \\ &+ \frac{A_{21}}{x-a_2} + \frac{A_{22}}{(x-a_2)^2} + \dots + \frac{A_{2k}}{(x-a_2)^{m_k}} + \\ &\dots + \frac{A_{k1}}{x-a_k} + \frac{A_{k2}}{(x-a_k)^2} + \dots + \frac{A_{kk}}{(x-a_k)^{m_k}}.\end{aligned}$$

Example 2.11. Find the partial fraction decomposition of

$$\frac{8}{(x+1)(x-1)^3}.$$

Solution: We have

$$\frac{8}{(x+1)(x-1)^3} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2} + \frac{D}{(x-1)^3}, \tag{2.10}$$

where A, B, C, D are some numbers. To find these numbers we bring the right-hand side to a common denominator, which gives

$$A(x-1)^3 + B(x+1)(x-1)^2 + C(x+1)(x-1) + D(x+1). \tag{2.11}$$

This polynomial must equal the polynomial in the numerator on the left-hand side of (2.10), i.e., equal 8 identically. By plugging in $x = 1$ and $x = -1$ we obtain respectively

$$2D = 8 \Longrightarrow D = 4,$$
$$A(-8) = 8 \Longrightarrow A = -1.$$

Thus, the polynomial in (2.11) becomes

$$-(x^3 - 3x^2 + 3x - 1) + B(x^3 - x^2 + x + 1) + C(x^2 - 1) + 4(x+1). \tag{2.12}$$

To find the other coefficients, we compute the coefficients of the term x^3 and the free term in (2.12), and equate them to the corresponding polynomial of the original fraction (which is the constant polynomial 8):

$$-x^3 + Bx^3 = 0 \Longrightarrow B = 1,$$
$$1 + 1 - C + 4 = 8 \Longrightarrow C = -2.$$

So finally we get

$$\frac{8}{(x+1)(x-1)^3} = -\frac{1}{x+1} + \frac{1}{x-1} - \frac{2}{(x-1)^2} + \frac{4}{(x-1)^3},$$

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Case 3. Factorization (2.8) contains linear and quadratic terms none of which appear more than once. In this case the following representation takes place:

$$\frac{P(x)}{Q(x)} = \frac{P(x)}{(x-a_1)\cdot\dots\cdot(x-a_k)(x^2+b_1x+c_1)\cdot\dots\cdot(x^2+b_lx+c_l)}$$
$$= \frac{A_1}{x-a_1} + \dots + \frac{A_k}{x-a_k} + \frac{B_1x+C_1}{x^2+b_1x+c_1} + \dots + \frac{B_lx+C_l}{x^2+b_lx+c_l}.$$

Example 2.12. Evaluate

$$\int \frac{6x^2+x-2}{2x^3-x-1}\,dx$$

We have $2x^3 - x - 1 = (x-1)(2x^2+2x+1)$, where the quadratic factor is irreducible. According to the above formula the appropriate partial fraction decomposition is

$$\frac{6x^2+x-2}{2x^3-x-1} = \frac{A}{x-1} + \frac{Bx+C}{2x^2+2x+1}.$$

By bringing the right-hand side to a common denominator and equating the numerators on both sides we get

$$6x^2 + x - 2 = A(2x^2+2x+1) + (Bx+C)(x-1)$$

Plugging $x = 1$ we get $A = 1$. Now we equate the coefficients of x^2 and the free terms on both sides:

$$6 = 2\cdot 1 + B, \quad -2 = 1 - C,$$

from which it follows that $B = 4$, $C = 3$. Therefore,

$$\int \frac{6x^2+x-2}{2x^3-x-1}dx = \int\left(\frac{1}{x-1} + \frac{4x+3}{2x^2+2x+1}\right) = \ln|x-1| + \int \frac{4x+3}{2x^2+2x+1}dx.$$

For the second integral we have

$$\int \frac{4x+3}{2x^2+2x+1}dx = \int \frac{4x+2}{2x^2+2x+1}dx + \int \frac{1}{2x^2+2x+1}dx$$

Here the first term on the right can be integrated using the substitution $y = 2x^2+2x+1$:

$$\int \frac{4x+2}{2x^2+2x+1}dx = \int \frac{dy}{y} = \ln(2x^2+2x+1),$$

while for the second term we observe that $2x^2+2x+1 = 2(x+\frac{1}{2})^2+\frac{1}{2}$ and so

$$\int \frac{dx}{2x^2+2x+1} = \int \frac{dx}{2(x+\frac{1}{2})^2+\frac{1}{2}} = 2\int \frac{dx}{4(x+\frac{1}{2})^2+1} = \tan^{-1}(2x+1),$$

where we used the substitution $y = 2(x+\frac{1}{2})$. Combining all the steps we obtain

$$\int \frac{6x^2+x-2}{2x^3-x-1}dx = \ln|x-1| + \ln(2x^2+2x+1) + \tan^{-1}(2x+1) + C.$$

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Case 4. Factorization (2.8) contains quadratic terms some of which appear more than once. This is, in fact, the most general case. The partial fraction decomposition will look similar to Cases 2 and 3 with the difference that any factor in the denominator of the form $(x^2+bx+c)^k$ will give the following terms:

$$\frac{1}{(x^2+bx+c)^k} = \frac{A_1x+B_1}{(x^2+bx+c)} + \frac{A_2x+B_2}{(x^2+bx+c)^2} + \cdots + \frac{A_kx+B_k}{(x^2+bx+c)^k}.$$

Example 2.13. Write down the general partial fraction decomposition of

$$\frac{x^3}{(x^2-2x+1)^2(x^2+2x+2)^2}.$$

Do not compute the values of the involved coefficients.

Solution: First observe that the first quadratic term is reducible: $(x^2-2x+1)^2 = (x-1)^4$. Therefore,

$$\begin{aligned}\frac{x^3}{(x^2-2x+1)^2(x^2+2x+2)^2} &= \frac{x^3}{(x-1)^4(x^2+2x+2)^2}\\ &= \frac{A_1}{x-1} + \frac{A_2}{(x-1)^2} + \frac{A_3}{(x-1)^3} + \frac{A_4}{(x-1)^4}\\ &+ \frac{B_1x+C_1}{x^2+2x+2} + \frac{B_2x+C_2}{(x^2+2x+2)^2}.\end{aligned}$$

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2.3. **Integration of Transcendental Functions.** In this section we provide examples of integration of trigonometric functions and radicals.

Example 2.14. To compute $\int \sin^2 x \cos^3 dx$ we use the substitution $t = \sin x$:

$$\int \sin^2 x \cos^3 dx = \int t^2(1-t^2)dt = \frac{t^3}{3} - \frac{t^5}{5} + C = \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C$$

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Example 2.15.
$$\int \sin^2 x\, dx = \int \frac{1}{2}(1-\cos 2x)dx = \frac{x}{2} - \frac{\sin 2x}{4} + C.$$
$\diamond$

Example 2.16.
$$\int \frac{dx}{\sin x} = \int \frac{\sin x\, dx}{\sin^2 x} = \Big| y = \cos x \Big| = \int \frac{dy}{y^2-1} = \frac{1}{2}\ln \frac{1-\cos x}{1+\cos x} + C,$$
where we used integration by partial fractions in the last step. $\diamond$

Example 2.17.
$$\int \sqrt{4-x^2}\, dx = \Big| x = 2\sin t; dx = 2\cos t\, dt \Big| = \int 2\cos t\ 2\cos t\, dt$$
$$= 2t + \sin 2t + C = 2t + 2\sin t\ \cos t + C = 2\sin^{-1}\frac{x}{2} + \frac{1}{2}x\sqrt{4-x^2} + C,$$
where the integration is done similarly to Example 2.15. $\diamond$

Exercises.

2· 1 Let
$$J_n = \int \frac{dx}{(x^2+1)^n}.$$
(i) Compute J_1.
(ii) Use recursive formula (2.6) to calculate J_4.

2· 2 Find a recursive formula analogous to (2.6) for
$$J_n = \int \frac{dx}{(x^2+a^2)^n}.$$
$n = 1, 2, \dots$.

2· 3 Let f be twice differentiable with $f(0) = 6$, $f(1) = 2$, and $f'(1) = 3$. Evaluate the integral $\int_0^1 x f''(x) dx$.

2· 4 Evaluate $\int 2t^2 \cos(t)\, dt$.

2· 5 Evaluate $\int \arcsin(4w)\, dw$.

2· 6 Find the integral
$\int (z+1)\, e^{6z} dz$.

2· 7 Find the integral $\int x^6 \ln(x)\ dx$.

2· 8 Find the integral $\int y\sqrt[4]{y+1}\, dy$.

2· 9 Find the integral $\int x^9 \sin(x^5)\, dx$.

2· 10 Find the integral $\int \frac{6\ln x}{x^2}\,dx$.

2· 11 Use integration by parts to evaluate the definite integral. $\int_1^e 6t^2 \ln t dt$.

2· 12 Evaluate the indefinite integral. $\int \ln(x^2+10x+25)\,dx$

2· 13 Evaluate $\displaystyle\int \frac{-x^2+33x+19}{(x+1)(x-4)(2x+1)}\,dx$

2· 14 What is the correct form of the partial fraction decomposition for the following integral?

$$\int \frac{x^2+1}{(x-2)^3(x^2+10x+38)}\,dx$$

(Do not evaluate the coefficients.)

2· 15 Evaluate the integral.

$$\int_{-1}^{1} \frac{x^3-3}{(x+4)(x+3)}dx.$$

2· 16 Write out the form of the partial fraction decomposition of the function appearing in the integral:

$$\int \frac{-4x-34}{x^2+3x-10}\,dx$$

Determine the numerical values of the coefficients A and B.

2· 17 Evaluate the integral $\int_4^5 \frac{3x-2}{x^2-2x+0}dx$

2· 18 Evaluate the integral. $\displaystyle\int_5^6 \frac{13x-19}{x^2-2x-3}\,dx = .$

2· 19 What is the correct form of the partial fraction decomposition for the following integral?

$$\int \frac{3(x^4+6)}{(x-7)(x^2-1)^2(x^2+7)^2}\,dx.$$

Do not evaluate coefficients.

2· 20 Make a substitution to express the integrand as a rational function and then evaluate the integral

$$\int \frac{-7}{\sqrt{x}-\sqrt[3]{x}}\,dx.$$

2· 21 If f is a quadratic function such that $f(0)=1$ and

$$\int \frac{f(x)}{x^2(x+1)^3}\,dx$$

is a rational function, find the value of $f'(0)$.

2· 22 Use integration by parts and the technique of partial fractions to evaluate the integral

$$\int -7\ln(x^2 - x + 2)\, dx.$$