v: 2019-04-04

REAL ANALYSIS LECTURE NOTES

RASUL SHAFIKOV

11. Fundamental solutions of differential operators

11.1. **Fundamental solutions.** In this section we study linear differential equations of the form

$$\sum_{|\alpha|\le m} a_\alpha D^\alpha u = f(x),\ \ f\in \mathcal{D}'(\mathbb{R}^n), \tag{1}$$

with constant coefficients $a_\alpha \in \mathbb{R}^n$. Define an order m linear differential operator

$$P(D) = \sum_{|\alpha|\le m} a_\alpha D^\alpha,\ \ a_\alpha \in \mathbb{R}^n.$$

Then the partial differential equation (1) takes the form

$$P(D)u = f(x),\ \ f\in \mathcal{D}'(\mathbb{R}^n). \tag{2}$$

Let Ω be a domain in $\mathbb{R}^n$. We say that $u \in \mathcal{D}'(\mathbb{R}^n)$ is a *generalized solution of (2) in* Ω if u satisfies this equation in Ω, that is,

$$\sum_{|\alpha|\le m} a_\alpha \langle D^\alpha u, \varphi\rangle = \langle f(x), \varphi\rangle$$

for every $\varphi \in \mathcal{D}'(\Omega)$.

Suppose that $f \in C(\Omega)$. If a function $u \in C^m(\Omega)$ satisfies (2), we call it *a classical solution* of (2). Obviously, if $u \in C^m(\Omega)$ is a generalized solution of (2), then it is a classical solution.

Definition 11.1. *A distribution $E \in \mathcal{D}'(\mathbb{R}^n)$ is called a fundamental solution of a differential operator $P(D)$ if*

$$P(D)E = \delta(x).$$

If u is a solution of the homogeneous equation $P(D)u = 0$ then $E + u$ also is a fundamental solution of (2), so in general a fundamental solution is not unique. The importance of this notion stems from the following statement.

Theorem 11.2. *Let $f \in \mathcal{D}'(\mathbb{R}^n)$ be a distribution such that the convolution*

$$u = E * f$$

exists in $\mathcal{D}'(\mathbb{R}^n)$. Then u is a solution of equation (2). Moreover, this solution of (2) is unique in the class of distributions in $\mathcal{D}'(\mathbb{R}^n)$ admitting the convolution with E.

Proof. Using the properties of convolution we obtain

$$P(D)(E*f) = \sum_{|\alpha|\le m} a_\alpha D^\alpha (E*f) = \left(\sum_{|\alpha|\le m} a_\alpha D^\alpha E\right) * f$$
$$= (P(D)E) * f = \delta * f = f.$$

Thus $u = E*f$ defines a solution of (2). In order to prove the uniqueness in the class of distributions, admitting the convolution with E, it suffices to prove that the homogeneous equation

$$P(D)v = 0$$

has a unique solution in this class. But this holds since

$$v = \delta * v = (P(D)E) * v = E * (P(D)v) = E * 0 = 0.$$

This proves the theorem. □

Example 11.1. Let $P(D) = \frac{d^2}{dx^2}$ on $\mathbb{R}$. To solve the equation

$$P(D)u = \chi_{[0,1]} \tag{3}$$

we first find a fundamental solution of the operator $P(D)$. If E satisfies $\frac{d^2E}{dx^2} = \delta$, then by Example 9.1, we have $\frac{dE}{dx} = \theta + c_1$. For convenience we may take $c_1 = -1/2$. Then $E = 1/2|x| + c_2$. Take $c_2 = 0$, then $E = 1/2|x|$ is a fundamental solution. The find a generalized solution of (3) we compute, according to Theorem 11.2, the convolution of the fundamental solution and the right-hand side of (3). Since one of the functions has compact support, the convolution is well-defined, so we have

$$E * \chi_{[0,1]}(x) = \int_{\mathbb{R}} \frac{1}{2}|y|\chi_{[0,1]}(x-y)dy = \frac{1}{2}\int_{\mathbb{R}} |x-t|\chi_{[0,1]}(t)dt = \frac{1}{2}\int_0^1 |x-t|dt.$$

This integral is a well defined C^1-smooth function on $\mathbb{R}$ given by

$$u(x) = \begin{cases} -\frac{x^2}{2} + \frac{1}{4}, & \text{if } x \leq 0, \\ \frac{x^2}{2} - \frac{x}{2} + \frac{1}{4}, & \text{if } 0 < x < 1, \\ \frac{x^2}{2} - \frac{1}{4}, & \text{if } x \geq 1. \end{cases}$$

◇

In the next section we compute fundamental solutions of the classical linear operators in $\mathbb{R}^n$.

11.2. Malgrange-Ehrenpreis theorem.

The following fundamental result is obtained independently by B. Malgrange and L. Ehrenpreis in 1954-55.

Theorem 11.3. *A linear differential operator with constant coefficients admits a fundamental solution in* $\mathcal{D}'(\mathbb{R}^n)$.

We will follow the proof given by J.-P. Rosay (Amer. Math. Monthly, 98 (1991), no. 6, p. 518–523.). In what follows it will be convenient to assume that all functions are complex valued. We denote by $||\cdot||$ the L^2-norm on $\mathbb{R}^n$, and

$$\langle \phi, \psi \rangle = \int_{\mathbb{R}^n} \phi\,\overline{\psi}$$

be the corresponding scalar product. If $P(D)$ is a linear differential operator with constant coefficients of order m, then its *adjoint* operator $P^*(D)$ is defined by the identity

$$\langle A\phi, \psi \rangle = \langle \phi, A^*\psi \rangle \quad \text{for all} \quad \phi, \psi \in L^2(\mathbb{R}^n).$$

In particular, if

$$P(D) = \sum_{|\alpha| \leq m} a_\alpha \frac{\partial^{|\alpha|}}{\partial x_\alpha}, \quad \alpha = (\alpha_1, \ldots, \alpha_n), \tag{4}$$

then the adjoint operator takes the form

$$P^*(D) = \sum_{|\alpha|\leq m} (-1)^{|\alpha|} \overline{a}_\alpha \frac{\partial^{|\alpha|}}{\partial x_\alpha}.$$

Proposition 11.4 (Hörmander's inequality)**.** *Let $P(D)$ be a nonzero linear differential operator with constant coefficients of order m given by (4). Then for every bounded domain $\Omega \subset \mathbb{R}^n$, there exists a constant $C > 0$, such that for every $\phi \in \mathcal{D}(\Omega)$, we have*

$$||P(D)\phi|| \geq C||\phi||.$$

One can take $C = |P|_m \, K_{m,\Omega}$, where

$$|P|_m = \max_{|\alpha|=m} \{|a_\alpha|\},$$

and $K_{m,\Omega}$ depends only on m and the quantity $\{\sup |x| : x \in \Omega\}$.

Proof. To illustrate the idea of the proof first consider the case $n = 1$, $\Omega = (0, 1)$, and $P(D) = d/dx$. We need to show that there exists some $C > 0$ such that $||\phi'|| \geq C||\phi||$ for all $\phi \in \mathcal{D}((0, 1))$. We have

$$\langle (x\phi)', \phi\rangle = \langle x\phi', \phi\rangle + \langle \phi, \phi\rangle.$$

Using integration by parts, $\langle (x\phi)', \phi\rangle = -\langle x\phi, \phi'\rangle$, and so $\langle \phi, \phi\rangle = -\langle x\phi', \phi\rangle - \langle x\phi, \phi'\rangle$. Since $|x| < 1$, we get $||\phi||^2 \leq 2||\phi'|| \, ||\phi||$, by the Hölder inequality (Thm 4.2). Hence, $||\phi'|| \geq 1/2||\phi||$.

The general case is proved by induction on the degree of P. Define a linear differential operator with constant coefficients $P_j(D)$ by the following identity

$$P(D)(x_j\phi) = x_jP(D)\phi + P_j(D)\phi.$$

The operator $P_j(D)$ is zero iff $P(D)$ does not involve any differentiation with respect to x_j. If it is nonzero, then $P_j(D)$ is of order at most $m - 1$. Let $A = \sup_{x\in\Omega} |x|$. By induction on m, we will show that for every $\phi \in \mathcal{D}(\Omega)$,

$$||P_j(D)\phi|| \leq 2mA \, ||P(D)\phi||. \tag{5}$$

Observe that (5) and the definition of P_j yield

$$||P(D)(x_j\phi)|| \leq (2m + 1)A \, ||P(D)\phi||. \tag{6}$$

Since differential operators with constant coefficients commute, we have for all $\phi \in \mathcal{D}(\Omega)$,

$$\begin{aligned} ||P(D)\phi||^2 = \langle P(D)\phi, P(D)\phi\rangle = \langle \phi, P^*(D)P(D)\phi\rangle &= \langle \phi, P(D)P^*(D)\phi\rangle \\ &= \langle P^*(D)\phi, P^*(D)\phi\rangle = ||P^*(D)\phi||^2. \end{aligned}$$

The inequality (5) is trivial for $m = 0$, since then $P_j(D) = 0$. Assuming that (5) is verified for operators of order $m - 1$, we compute $\langle P(D)(x_j\phi), P_j(D)\phi\rangle$ in two different ways. From the definition of $P_j(D)$ we have,

$$\langle P(D)(x_j\phi), P_j(D)\phi\rangle = \langle x_jP(D)\phi, P_j(D)\phi\rangle + ||P_j(D)\phi||^2.$$

By integration by parts (i.e., using the definition of the adjoint) and using commutativity of $P^*(D)$ and $P_j(D)$, we obtain

$$\langle P(D)(x_j\phi), P_j(D)\phi\rangle = \langle P_j^*(D)(x_j\phi), P^*(D)\phi\rangle.$$

Therefore,

$$||P_j(D)\phi||^2 = \langle P_j^*(D)(x_j\phi), P^*(D)\phi\rangle - \langle x_jP(D)\phi, P_j(D)\phi\rangle. \tag{7}$$

By the induction hypothesis, equation (6) holds for all operators of order $m-1$, which when applied to $P_j^*(D)$ yields

$$||P_j^*(D)(x_j\phi)|| \leq (2m-1)A\,||P_j(D)\phi||.$$

And, since

$$|\langle x_j P(D)\phi, P_j(D)\phi\rangle| \leq A||P(D)\phi||\,||P_j(D)\phi||,$$

we obtain from (7) that

$$||P_j(D)\phi||^2 \leq 2mA\,||P_j(D)\phi||\,||P(D)\phi||,$$

which proves (5). If $P(D)$ is an operator of order $m \geq 1$, there exists $j \in \{1,\dots,n\}$ such that $P_j(D)$ is of order $m-1$, and $|P_j|_{m-1} \geq |P|_m$. Thus the proposition follows from (5) by induction on m. □

Corollary 11.5. *If Ω is a bounded domain in $\mathbb{R}^n$, then for every $g \in L^2(\Omega)$ there exists $u \in L^2(\Omega)$ such that $P(D)u = g$.*

Proof. This follows from the inequality $||P^*(D)\phi|| \geq C||\phi||$, $\phi \in \mathcal{D}(\Omega)$. Indeed, $P(D)u = g$ means that for all $\phi \in \mathcal{D}(\Omega)$,

$$\langle g, \phi\rangle = \langle u, P^*(D)\phi\rangle. \tag{8}$$

Let

$$E = \{\psi \in \mathcal{D}(\Omega),\ \psi = P^*(D)\phi \text{ for some } \phi \in \mathcal{D}(\Omega)\}.$$

Consider the (anti)linear functional $l : E \to \mathbb{C}$ given by

$$l(\psi) = \langle g, \phi\rangle, \quad \text{where} \quad \psi = P^*(D)\phi.$$

Then using Hörmander's inequality we have

$$||l|| = \sup_{||\psi||=1} |\langle g,\phi\rangle| \leq ||g|| \sup_{||\psi||=1} ||\phi|| \leq \frac{||g||}{C} \sup_{||\psi||=1} ||P^*(D)\phi|| = \frac{||g||}{C}.$$

This shows that l is a bounded linear functional on E with L^2-norm. Therefore, l can be extended to $\overline{E}$, the closure of E in $L^2(\Omega)$. Then the Riesz representation theorem (Theorem 4.11) gives the existence of $u \in \overline{E}$ such that $l(\psi) =< u, \psi >$. This implies equation (8). □

We now wish to extend the above result to $L^2_{loc}(\Omega)$ functions. For this we first prove the following

Proposition 11.6. *There exists $C' > 0$ such that for all $\eta \in \mathbb{R}$ and $\phi \in \mathcal{D}(\Omega)$, we have*

$$\int_\Omega e^{\eta x_1}\,|P(D)\phi|^2 \geq C' \int_\Omega e^{\eta x_1}\,|\phi|^2.$$

Note that C' is independent of η.

Proof. Apply Hörmander's inequality to $\Psi = e^{(\eta/2)x_1}\,\phi$ and operator $Q(D)$ defined by

$$Q(D)(\Psi) = e^{(\eta/2)x_1} P(D)[e^{-(\eta/2)x_1}\,\Psi],$$

which is indeed a constant coefficient operator of the same degree m as $P(D)$. □

Corollary 11.7. *Let $\phi \in \mathcal{D}(\mathbb{R}^n)$ or more generally $\phi \in L^2(\mathbb{R}^n)$ with compact support. If $P(D)\phi$ is supported in the ball $B(0,r)$, then so is ϕ.*

Proof. By letting $\eta \to +\infty$ in Proposition 11.6, one can immediately verify that if $P(D)\phi = 0$ in the half-space $\{x_1 > 0\}$, then $\phi = 0$ there. From this, using translations and rotations, the corollary can be verified in the case of a smooth ϕ. In the nonsmooth case, for $\varepsilon < 1$ consider the regularization $\phi_\varepsilon = \phi * \omega_e \in \mathcal{D}(\mathbb{R}^n)$. Then $P(D)\phi_\varepsilon = P(D)\phi * \omega_\varepsilon$ is supported in $B(0, r+\varepsilon)$ and $\phi_\varepsilon \to \phi$ in L^2 as $\varepsilon \to 0$ by Proposition 10.8. This reduces the problem to the smooth case. □

Proposition 11.8. *Let $0 < r < r' < R$. If $v \in L^2(B(0,r'))$ and satisfy $P(D)v = 0$ on $B(0,r')$, then there exists a sequence $(v_j) \subset L^2(B(0,R))$ such that $P(D)v_j = 0$ on $B(0,R)$ and $v_j \to v$ in $L^2(B(0,r))$ as $j \to \infty$.*

Proof. After regularization we can assume that v is smooth, possibly shrinking r' slightly. It suffices to show that any continuous linear functional that vanishes on the space $L^2(B(0,R)) \cap \{\alpha : P(D)\alpha = 0\}$ also vanishes at v. In other words (using the Riesz representation theorem), we have to show that if $g \in L^2(B(0,r))$ and satisfies $\langle \alpha, g\rangle_{B(0,r)} = 0$ for all $\alpha \in L^2(B(0,R))$ with $P(D)\alpha = 0$, then $\langle v, g\rangle_{B(0,r)} = 0$.

Claim. There exists $w \in L^2(B(0,R))$ such that for all $\phi \in \mathcal{D}(\mathbb{R}^n)$,

$$\langle \phi, g\rangle_{B(0,r)} = \langle P(D)\phi, w\rangle_{B(0,R)}.$$

For the proof of the claim, we need to find $C > 0$ such that

$$\left|\langle \phi, g\rangle_{B(0,r)}\right| \leq C||P(D)\phi||_{B(0,R)}.$$

Notice that if $P(D)\phi = 0$, then we have $\langle \phi, g\rangle = 0$. If $P(D)\phi \neq 0$, then by Corollary 11.5 we can find $\Psi \in L^2(B(0,R))$ so that $P(D)\Psi = P(D)\phi$ and $||\Psi||_{B(0,R)} \leq C_1||P(D)\phi||_{B(0,R)}$ for some $C_1 > 0$. Then

$$\langle \phi, g\rangle_{B(0,R)} = \langle \phi - \Psi, g\rangle_{B(0,R)} + \langle \Psi, g\rangle_{B(0,R)} = \langle \Psi, g\rangle_{B(0,R)}.$$

Hence, $|\langle \phi, g\rangle_{B(0,R)}| \leq C||P(D)\phi||_{B(0,R)}$ with $C = C_1||g||$, which proves the claim.

Pick w as given by the claim. Extend g and w on $\mathbb{R}^n$ to $\tilde{g}$ and $\tilde{w}$ by setting $\tilde{g} = 0$ on $\mathbb{R}^n \setminus B(0,r)$ and $\tilde{w} = 0$ on $\mathbb{R}^n \setminus B(0,R)$. We then have $\tilde{g} = P^*(D)\tilde{w}$. Since $\tilde{w}$ has compact support, and $P^*(D)\tilde{w}$ is supported in $B(0,r)$, we conclude from Corollary 11.7 that $w = 0$ on $B(0,R) \setminus B(0,r)$.

To complete the proof of the proposition take v as at the beginning of the proof, and extend it to be a smooth, compactly supported function on $\mathbb{R}^n$ (but no longer satisfying $P(D)v = 0$ off $B(0,r)$). One has

$$\langle v, g\rangle_{B(0,r)} = \langle P(D)v, w\rangle_{B(0,R)} = \langle P(D)v, w\rangle_{B(0,r)} = 0.$$

□

We now can prove the following

Proposition 11.9. *Let $P(D)$ be a nonzero linear differential operator on $\mathbb{R}^n$ with constant coefficients. Then for every $g \in L^2_{loc}(\mathbb{R}^n)$ there exists $u \in L^2_{loc}(\mathbb{R}^n)$ such that $P(D)u = g$.*

Proof. By Corollary 11.5 there exists $u_1 \in L^2(B(0,2))$ so that $P(D)u_1 = g$ on $B(0,2)$. Then inductively, assuming u_p has been chosen in $L^2(B(0,p+1))$ so that $P(D)u_p = g$, one chooses u_{p+1} in $L^2(B(0,p+2))$ in the following way. Let w be an arbitrary solution of $P(D)w = g$, in $L^2(B(0,p+2))$. On $B(0,p+1)$ one has $P(D)(u_p - w) = 0$. By Proposition 11.8 there exists $v \in L^2(B(0,p+2))$ such that $P(D)v = 0$, and $||v - (u_p - w)||_{B(0,p)} \leq 1/2^p$. Set $u_{p+1} = v + w$. Then $P(D)u_{p+1} = g$ on $B(0,p+2)$, and $||u_{p+1} - u_p||_{B(0,p)} \leq 1/2^p$. The sequence (u_p) is obviously convergent in $L^2_{loc}(\mathbb{R}^n)$, and its limit satisfies $P(D)u = g$. □

Proof of the Malgrange-Ehrenpreis Theorem. Let H be the function (product of the Heaviside functions on $\mathbb{R}$) defined on $\mathbb{R}^n$ by

$$H(x_1, \ldots, x_n) = \begin{cases} 1, & \text{if } x_j > 0,\ j = 1, \ldots, n, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\frac{\partial^n H}{\partial x_1 \dots \partial x_n} = \delta_0.$$

Since $H \in L^2_{loc}(\mathbb{R}^n)$, by the previous proposition there exists $u \in L^2_{loc}(\mathbb{R}^n)$ so that $P(D)u = H$. Set

$$E = \frac{\partial^n u}{\partial x_1 \dots \partial x_n}.$$

Then

$$P(D)E = P(D)\frac{\partial^n u}{\partial x_1 \dots \partial x_n} = \frac{\partial^n}{\partial x_1 \dots \partial x_n}(P(D)u) = \delta_0.$$

□