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REAL ANALYSIS LECTURE NOTES

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12. Fundamental solutions of classical operators.

12.1. **More advanced examples.** Here we consider examples concerning distributions in $\mathbb{R}^n$, $n > 1$. One of the most important examples is given by the Cauchy-Riemann operator $\frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial \overline{x}} + i\frac{\partial}{\partial \overline{y}}\right)$ on the complex plane $\mathbb{C} \cong \mathbb{R}^2$ with the coordinate $z = x + iy$. This is a differential operator with constant coefficients of order one.

First of all we adapt the integration by parts (formula (12) in Section 5.2) to the complex notation. Let Ω be a bounded domain with C^1 boundary in $\mathbb{C}$ and f be a complex function of class $C(\overline{\Omega})$. We suppose that (a connected component of) $\partial\Omega$ is positively parametrized by the map $[a, b] \ni t \mapsto x(t) + iy(t)$ of class C^1. Then

$$\vec{n} = \frac{(y'(t), -x'(t))}{\sqrt{(x'(t))^2 + (y'(t))^2}}$$

is the vector field of the unit outward normal. Then, from the definition of the surface integral (see Section 5.1) and using the notation $dz = dx + idy$, we have

$$\int_{\partial\Omega} f[(\vec{n}, \vec{e}_1) + i(\vec{n}, \vec{e}_2)]dS = \int_a^b f(x(t), y(t))(y'(t) - ix'(t))dt = -i\int_{\partial\Omega} f(z)dz.$$

Keeping this in mind, we pass to the integration by parts with the Cauchy-Riemann operator. For two complex-valued functions $u, v \in C^1(\overline{\Omega})$ we have

$$\int_\Omega \frac{\partial u}{\partial \overline{z}} v dxdy = \frac{1}{2}\int_\Omega \frac{\partial u}{\partial x} v dxdy + \frac{i}{2}\int_\Omega \frac{\partial u}{\partial y} v dxdy = \frac{1}{2}\left(\int_{\partial\Omega} uv(\vec{n}, e_1)dS - \int_\Omega u\frac{\partial v}{\partial x}dxdy\right)$$
$$+\frac{i}{2}\left(\int_{\partial\Omega} uv(\vec{n}, e_2)dS - \int_\Omega u\frac{\partial v}{\partial y}dxdy\right) = \frac{1}{2}\int_{\partial\Omega} uv[(\vec{n}, e_1) + i(\vec{n}, e_2)]dS - \int_\Omega u\frac{\partial v}{\partial \overline{z}}dxdy$$
$$= \frac{-i}{2}\int_{\partial\Omega} uvdz - \int_\Omega u\frac{\partial v}{\partial \overline{z}}dxdy.$$

Thus we obtain the following useful integration by parts formula:

$$\int_\Omega \frac{\partial u}{\partial \overline{z}} v dxdy = \frac{-i}{2}\int_{\partial\Omega} uvdz - \int_\Omega u\frac{\partial v}{\partial \overline{z}}dxdy. \tag{1}$$

Lemma 12.1. *The function $\frac{1}{\pi z}$ is the fundamental solution of the operator $\frac{\partial}{\partial \overline{z}}$, i.e.,*

$$\frac{\partial}{\partial \overline{z}}\frac{1}{z} = \pi\delta(x, y). \tag{2}$$

Proof. First note that $\frac{1}{z} \in L^1_{loc}(\mathbb{R}^2)$ (use the polar coordinates to verify this), and so $\frac{1}{z}$ defines a regular distribution. Let $\varphi \in \mathcal{D}(\mathbb{R}^2)$ be a (complex-valued) test function with $\operatorname{supp}\varphi \subset \mathrm{B}(0, \mathrm{R})$.

For $\varepsilon > 0$ denote by $A(\varepsilon, R)$ the annulus $B(0,R)\backslash\overline{B(0,\varepsilon)}$. Denote also by C_ε the circle $\{|z| = \varepsilon\}$. Then $\frac{\partial}{\partial \overline{z}}\frac{1}{z} = 0$ on $A(\varepsilon, R)$ and using (1) with $u = \phi$ and $v = 1/z$ we have

$$\langle \frac{\partial}{\partial \overline{z}}\frac{1}{z}, \varphi\rangle = -\langle \frac{1}{z}, \frac{\partial \varphi}{\partial \overline{z}}\rangle = -\lim_{\varepsilon \longrightarrow 0+}\int_{\Omega_\varepsilon}\frac{1}{z}\frac{\partial\varphi}{\partial\overline{z}}dxdy = \lim_{\varepsilon\longrightarrow 0+} -\frac{i}{2}\int_{C_\varepsilon}\frac{\varphi}{z}dz.$$

Here the integral over the circle C_ε is taken with positive orientation with respect to the disc $B(0,\varepsilon)$. Writing

$$\int_{C_\varepsilon}\frac{\varphi}{z}dz = \int_{C_\varepsilon}\frac{\varphi(z)-\varphi(0)}{z}dz + \varphi(0)\int_{C_\varepsilon}\frac{dz}{z},$$

we easily see that the first integral tends to 0 (use Taylor's formula) and the second one tends to $2\pi i\varphi(0)$. Hence,

$$\lim_{\varepsilon\longrightarrow 0+} -\frac{i}{2}\int_{C_\varepsilon}\frac{\varphi}{z}dz = \pi\varphi(0),$$

which concludes the proof. □

Using Lemma 12.1 we can easily deduce an integral representation involving the Cauchy-Riemann operator. Fix $z \in \Omega$. Denote by Ω_ε the domain $\Omega\backslash B(z,\varepsilon)$ and by $C(z,\varepsilon)$ the circle $\{\zeta : |\zeta - z| = \varepsilon\}$. Let a complex function f be of class $C^1(\overline{\Omega})$. We set $\zeta = \xi + i\eta$. Then, using (1) and (2), we have

$$\frac{1}{\pi}\int_\Omega \frac{\partial f(\zeta)}{\partial\overline{\zeta}}\frac{1}{\zeta - z}d\xi d\eta = \lim_{\varepsilon\longrightarrow 0+}\int_{\Omega_\varepsilon}\frac{\partial f(\zeta)}{\partial\overline{\zeta}}\frac{1}{\zeta - z}d\xi d\eta$$
$$= \frac{1}{\pi}\lim_{\varepsilon\longrightarrow 0+}\left(\frac{-i}{2}\int_{\partial\Omega}\frac{f(\zeta)}{\zeta - z}d\zeta + \frac{i}{2}\int_{C(z,\varepsilon)}\frac{f(\zeta)}{\zeta - z}d\zeta\right) = \frac{-i}{2\pi}\int_{\partial\Omega}\frac{f(\zeta)}{\zeta - z}d\zeta - f(z).$$

Thus we obtained the so-called *Cauchy-Green formula*

$$f(z) = \frac{1}{2\pi i}\int_{\partial\Omega}\frac{f(\zeta)}{\zeta - z}d\zeta - \frac{1}{\pi}\int_\Omega\frac{\partial f(\zeta)}{\partial\overline{\zeta}}\frac{1}{\zeta - z}d\xi d\eta. \tag{3}$$

In particular, if f is holomorphic, i.e., $\frac{\partial f(\zeta)}{\partial\overline{\zeta}} = 0$ in Ω, we have the classical *Cauchy integral formula*

$$f(z) = \frac{1}{2\pi i}\int_{\partial\Omega}\frac{f(\zeta)}{\zeta - z}d\zeta. \tag{4}$$

It is also easy to deduce now the *Cauchy theorem.* Let D be a domain in $\mathbb{C}$ and γ be a closed simple path in D homotopic to 0. If f is a holomorphic function in D, then

$$\int_\gamma f(z)dz = 0. \tag{5}$$

Indeed, consider the domain $\Omega \subset D$ bounded by γ. Since γ is homotopic to 0, the boundary $\partial\Omega$ of Ω coincides with γ (with suitable orientation). Fix $z \in \Omega$. By the Cauchy formula we have

$$\int_\gamma f(\zeta)d\zeta = \int_\gamma\frac{(\zeta - z)f(\zeta)}{\zeta - z}d\zeta = \int_\gamma\frac{\zeta f(\zeta)}{\zeta - z}d\zeta - z\int_\gamma\frac{f(\zeta)}{\zeta - z}d\zeta$$
$$= 2\pi i z f(z) - 2\pi i z f(z) = 0,$$

which proves (5).

The next example is a generalization of Example (9.2).

Example 12.1. Let Ω be a bounded domain with C^1 boundary and $f \in C^1(\overline{\Omega}) \cap C^1(\mathbb{R}^n \backslash \Omega)$ (in particular, all discontinuity points of f belong to $\partial\Omega$). The usual partial derivative $\frac{\partial f}{\partial x_k}$ is defined and locally integrable on $\mathbb{R}^n \backslash \partial\Omega$ so we can consider the regular distribution $T_{\frac{\partial f}{\partial x_k}} \in \mathcal{D}'(\mathbb{R}^n)$. We also introduce the "jump" of f on $\partial\Omega$:

$$[f]_{\partial\Omega}(x) = f_+(x) - f_-(x) = \lim_{\mathbb{R}^n \backslash \overline{\Omega} \ni x' \longrightarrow x} f(x') - \lim_{\Omega \ni x' \longrightarrow x} f(x'),\ x \in \partial\Omega.$$

We point out here that if μ is a continuous function on a compact hypersurface $\Gamma \subset \mathbb{R}^n$, then the distribution $\mu\delta_\Gamma$ defined by

$$\langle \mu\delta_\Gamma, \varphi \rangle = \int_\Gamma \mu\, \varphi\, dS, \quad \varphi \in \mathcal{D}(\mathbb{R}^n)$$

is called *the simple potential on the hypersurface* Γ *with density* μ.

For $k = 1, 2, \dots, n$, consider the distribution $[f]_{\partial\Omega}(e_k, \vec{n})\delta_{\partial\Omega} \in \mathcal{D}'(\mathbb{R}^n)$ defined by

$$\langle [f]_{\partial\Omega}\,(e_k, \vec{n})\,\delta_{\partial\Omega}, \varphi \rangle = \int_{\partial\Omega} [f]_{\partial\Omega}\,(e_k, \vec{n})\,\varphi\, dS, \quad \varphi \in \mathcal{D}(\mathbb{R}^n).$$

Let us prove the formula for the partial derivative of f in the sense of distributions:

$$\frac{\partial f}{\partial x_k} = T_{\frac{\partial f}{\partial x_k}} + [f]_{\partial\Omega}(e_k, \vec{n})\delta_{\partial\Omega}, \tag{6}$$

where $\frac{\partial f}{\partial x_k} \in \mathcal{D}'(\mathbb{R}^n)$. We have

$$\langle \frac{\partial f}{\partial x_k}, \varphi \rangle = -\langle f, \frac{\partial \varphi}{\partial x_k} \rangle = -\int_{\mathbb{R}^n} f(x) \frac{\partial \varphi(x)}{\partial x_k} dx.$$

We decompose

$$\int_{\mathbb{R}^n} f(x) \frac{\partial \varphi(x)}{\partial x_k} dx = \int_\Omega f(x) \frac{\partial \varphi(x)}{\partial x_k} dx + \int_{\mathbb{R}^n \backslash \Omega} f(x) \frac{\partial \varphi(x)}{\partial x_k} dx$$

and apply to every integral on the right the integration by parts formula. Then

$$\int_\Omega f(x) \frac{\partial \varphi(x)}{\partial x_k} dx = -\int_\Omega \varphi(x) \frac{\partial f(x)}{\partial x_k} dx + \int_{\partial\Omega} f_-(x)\varphi(x)(e_k, \vec{n}) dS,$$

and

$$\int_{\mathbb{R}^n \backslash \Omega} f(x) \frac{\partial \varphi(x)}{\partial x_k} dx = -\int_{\mathbb{R}^n \backslash \Omega} \varphi(x) \frac{\partial f(x)}{\partial x_k} dx - \int_{\partial\Omega} f_+(x)\varphi(x)(e_k, \vec{n}) dS$$

(the minus sign before the last integral appears because $\vec{n}$ is the exterior normal for Ω and so it is the interior normal for $\mathbb{R}^n \backslash \Omega$). Therefore,

$$\int_{\mathbb{R}^n} f(x) \frac{\partial \varphi(x)}{\partial x_k} dx = -\int_{\mathbb{R}^n} \varphi(x) \frac{\partial f(x)}{\partial x_k} dx - \int_{\partial\Omega} [f]_{\partial\Omega}(x)(e_k, \vec{n})\varphi(x) dS,$$

and

$$\langle \frac{\partial f}{\partial x_k}, \varphi \rangle = -\int_{\mathbb{R}^n} \varphi(x) \frac{\partial f(x)}{\partial x_k} dx + \int_{\partial\Omega} [f]_{\partial\Omega}(x)(e_k, \vec{n})\varphi(x) dS,$$

which proves (6). $\diamond$

12.2. **Laplace operator.** In this section we construct a fundamental solution of the Laplace operator

$$\Delta = \frac{\partial^2}{\partial x_1^2} + ... + \frac{\partial^2}{\partial x_n^2}.$$

(a) First we suppose that $n = 2$ and prove that

$$\Delta \ln |x| = \frac{\partial^2 \ln |x|}{\partial x_1^2} + \frac{\partial^2 \ln |x|}{\partial x_2^2} = 2\pi\delta(x), \quad x \in \mathbb{R}^2. \tag{7}$$

First of all observe that the function $\ln |x|$ is of class $L^1_{loc}(\mathbb{R}^2)$ (to see this it suffices to pass to the polar coordinates) and so it can be viewed as a distribution. Let $\varphi \in \mathcal{D}(\mathbb{R}^2)$. Since $\operatorname{supp} \varphi$ is a compact set, there exists $R > 0$ such that $\varphi(x) = 0$ for $|x| \geq R/2$. We have

$$\langle \Delta \ln |x|, \varphi \rangle = \langle \ln |x|, \Delta\varphi \rangle = \int_{\mathbb{R}^2} \ln |x| \Delta\varphi(x) dx = \int_{|x| \leq R} \ln |x| \Delta\varphi(x) dx.$$

Denote by $A(\varepsilon, R) = \{x : \varepsilon < |x| < R\}$ the annulus, where $\varepsilon > 0$ is small enough. Then by the Lebesgue convergence theorem,

$$\int_{|x| \leq R} \ln |x| \Delta\varphi(x) dx = \lim_{\varepsilon \longrightarrow 0+} \int_{A(\varepsilon,R)} \ln |x| \Delta\varphi(x) dx.$$

By the Green formula we have

$$\int_{A(\varepsilon,R)} \ln |x| \Delta\varphi(x) dx = \int_{A(\varepsilon,R)} \Delta \ln |x| \varphi(x) dx + \int_{\partial A(\varepsilon,R)} \left(\ln |x| \frac{\partial \varphi(x)}{\partial \vec{n}} - \varphi \frac{\partial \ln |x|}{\partial \vec{n}} \right) dS.$$

An elementary computation (say, in the polar coordinates) shows that $\Delta \ln |x| = 0$ for $x \neq 0$ so the first integral on the right vanishes. Furthermore, $\partial A(\varepsilon, R) = C_\varepsilon \cup C_R$, where $C_r = \{x : |x| = r\}$ so that $\int_{\partial A(\varepsilon,R)} = \int_{C_\varepsilon} + \int_{C_R}$. By the choice of R we have $\varphi(x) = \frac{\varphi(x)}{\partial \vec{n}} = 0$ for $x \in C_R$. Thus,

$$\int_{A(\varepsilon,R)} \ln |x| \Delta\varphi(x) dx = \int_{C_\varepsilon} \left(\ln |x| \frac{\partial \varphi(x)}{\partial \vec{n}} - \varphi \frac{\partial \ln |x|}{\partial \vec{n}} \right) dS.$$

Since $\vec{n}$ is the vector of the unit exterior normal to $A(\varepsilon, R)$, for every $x \in C_\varepsilon$ we have $\vec{n} = -x/|x|$, and so

$$\frac{\partial}{\partial \vec{n}} = -\frac{\partial}{\partial \vec{x}_1} \frac{x_1}{|x|} - \frac{\partial}{\partial \vec{x}_2} \frac{x_2}{|x|}.$$

Then,

$$\left| \int_{C_\varepsilon} \ln |x| \frac{\partial \varphi(x)}{\partial \vec{n}} dS \right| \leq const \cdot \varepsilon |\ln \varepsilon| \longrightarrow 0, \ \varepsilon \longrightarrow 0.$$

Finally, $\frac{\partial \ln |x|}{\partial \vec{n}} = -\frac{1}{|x|}$ so that

$$-\int_{C_\varepsilon} \varphi \frac{\partial \ln |x|}{\partial \vec{n}} dS = \frac{1}{\varepsilon} \int_{C_\varepsilon} \varphi dS.$$

But we have

$$\lim_{\varepsilon \longrightarrow 0} \frac{1}{\varepsilon} \int_{C_\varepsilon} \varphi(x) dS = \left(\lim_{\varepsilon \longrightarrow 0} \frac{1}{\varepsilon} \int_{C_\varepsilon} (\varphi(x) - \varphi(0)) dS \right) + 2\pi\varphi(0) = 2\pi\varphi(0).$$

Thus,

$$\int_{\mathbb{R}^2} \ln |x| \Delta\varphi(x) dx = 2\pi\varphi(0),$$

which proves (7).

(b) Now we show that

$$\Delta \frac{1}{|x|^{n-2}} = -(n-2)S_n\delta(x), \quad n \geq 3, \tag{8}$$

where the constant S_n is equal to the surface area of the unit sphere in $\mathbb{R}^n$. The proof is quite similar to part (a). We use the notation $r = r(x) = |x|$. For a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ of class C^2 we have

$$\frac{\partial}{\partial x_j} f(r) = f'(r)\frac{x_j}{r},$$

$$\frac{\partial^2}{\partial x_j^2} f(r) = f''(r)\frac{x_j^2}{r^2} + f'(r)\frac{r^2 - x_j^2}{r^2},$$

$$\Delta f(r) = f''(r) + f'(r)\frac{n-1}{r}.$$

Setting $f(r) = r^p$, we obtain

$$\Delta r^p = p(p+n-2)r^{p-2}.$$

Therefore, $\Delta r^{2-n} = 0$ on $\mathbb{R}^n\backslash\{0\}$. Also note that the function $x \mapsto r^{2-n}$ is in $L^1_{loc}(\mathbb{R}^n)$.

We have, for sufficiently large $R > 0$, that

$$\langle \Delta r^{2-n}, \varphi\rangle = \langle r^{2-n}, \Delta\varphi\rangle = \int_{\mathbb{R}^n} r^{2-n}\Delta\varphi(x) = \int_{|x|\leq R} r^{2-n}\Delta\varphi(x)dx,$$

and

$$\int_{|x|\leq R} r^{2-n}\Delta\varphi(x)dx = \lim_{\varepsilon\longrightarrow 0+}\int_{A(\varepsilon,R)} r^{2-n}\Delta\varphi(x)dx.$$

Again, by the Green formula we have

$$\int_{A(\varepsilon,R)} r^{2-n}\Delta\varphi(x)dx = \int_{A(\varepsilon,R)} \Delta r^{2-n}\varphi(x)dx + \int_{\partial A(\varepsilon,R)}\left(r^{2-n}\frac{\partial\varphi(x)}{\partial\vec{n}} - \varphi\frac{\partial r^{2-n}}{\partial\vec{n}}\right)dS.$$

The first integral on the right vanishes and by the choice of R we have $\varphi(x) = \frac{\varphi(x)}{\partial\vec{n}} = 0$ for $x \in C_R$. Thus,

$$\int_{A(\varepsilon,R)} r^{2-n}\Delta\varphi(x)dx = \int_{C_\varepsilon}\left(r^{2-n}\frac{\partial\varphi(x)}{\partial\vec{n}} - \varphi\frac{\partial r^{2-n}}{\partial\vec{n}}\right)dS.$$

Since $\vec{n}$ is the vector of the unit exterior normal to $A(\varepsilon,R)$, for every $x \in C_\varepsilon$ we have $\vec{n} = -x/|x|$ and

$$\frac{\partial}{\partial\vec{n}} = -\frac{\partial}{\partial\vec{x}_1}\frac{x_1}{|x|} - \cdots - \frac{\partial}{\partial\vec{x}_n}\frac{x_n}{|x|}.$$

Then,

$$\left|\int_{C_\varepsilon} r^{2-n}\frac{\partial\varphi(x)}{\partial\vec{n}}dS\right| \leq const\cdot\varepsilon \longrightarrow 0, \quad \varepsilon\to 0.$$

Finally, $\frac{\partial r^{2-n}}{\partial\vec{n}} = (n-2)r^{1-n}$, so that

$$-\int_{C_\varepsilon}\varphi\frac{\partial r^{2-n}}{\partial\vec{n}}dS = -(n-2)\frac{1}{\varepsilon^{n-1}}\int_{C_\varepsilon}\varphi dS.$$

Then

$$-(n-2)\lim_{\varepsilon\longrightarrow 0}\frac{1}{\varepsilon^{n-1}}\int_{C_\varepsilon}\varphi(x)dS = -(n-2)\left(\lim_{\varepsilon\longrightarrow 0}\frac{1}{\varepsilon^{n-1}}\int_{C_\varepsilon}(\varphi(x)-\varphi(0))dS\right)$$
$$-(n-2)S_n\varphi(0) = -(n-2)S_n\varphi(0),$$

which proves (8).

If $n=3$, then $S_3 = 4\pi$ so that

$$\Delta\frac{1}{r} = -4\pi\delta(x), \quad x\in\mathbb{R}^3.$$

12.3. **Heat Equation.** Consider the function

$$E(x,t) = \frac{\theta(t)}{(2a\sqrt{\pi t})^n}e^{-\frac{|x|^2}{4a^2t}},$$

where the function θ is the Heaviside function on $\mathbb{R}$. The function E is locally integrable in $\mathbb{R}^{n+1}$. Indeed, $E(x,t)=0$ if $t<0$ and $E(x,t)$ is positive for $t\geq 0$. Furhermore, E is continuous (and vanishes) on the hyperplane $\{(x,t) : t=0\}$. Consider a bouded set of $\mathbb{R}^{n+1}$ of the form $B(0,R)\times[0,R]$, where $B(0,R)=\{x\in\mathbb{R}^n : |x|\leq R\}$. By Fubini's theorem we have

$$\int_{B(0,R)\times[0,R]}E(x,t)dxdt = \int_{[0,R]}\left(\int_{B(0,R)}E(x,t)dx\right)dt \leq \int_{[0,R]}\left(\int_{\mathbb{R}^n}E(x,t)dx\right)dt.$$

After the change of coordinates $x/2a\sqrt{t}=y$ we have

$$\int_{\mathbb{R}^n}E(x,t)dx = \int_{\mathbb{R}^n}\frac{1}{(2a\sqrt{\pi t})^n}e^{-\frac{|x|^2}{4a^2t}}dx = \frac{1}{(\sqrt{\pi})^n}\prod_{j=1}^{n}\int_{\mathbb{R}}e^{-y_j^2}dy_j = 1.$$

Thus,

$$\int_{\mathbb{R}^n}E(x,t)dx = 1, \tag{9}$$

and so

$$\int_{[0,R]}\left(\int_{\mathbb{R}^n}E(x,t)dx\right)dt \leq \int_{[0,R]}dt = R.$$

This proves the local integrability of $E(x,t)$.

Let us prove the following identity:

$$\frac{\partial E}{\partial t} - a^2\Delta E = \delta(x,t). \tag{10}$$

We first observe that for $t>0$ the function E is of class C^∞, and by an elementary computation, which is left for the reader, we have

$$\frac{\partial E}{\partial t}(x,t) - a^2\Delta E = 0,\ t>0. \tag{11}$$

Here the derivatives are taken in the usual sense. Now let $\varphi\in\mathcal{D}(\mathbb{R}^{n+1})$. Then

$$\langle\frac{\partial E}{\partial t} - a^2\Delta E,\varphi\rangle = -\langle E,\frac{\partial\varphi}{\partial t}+a^2\Delta\varphi\rangle = -\int_0^\infty\left(\int_{\mathbb{R}^n}E(x,t)\left(\frac{\partial\varphi}{\partial t}+a^2\Delta\varphi\right)dx\right)dt.$$

By the Lebesgue convergence theorem we have

$$-\int_0^\infty\left(\int_{\mathbb{R}^n}E(x,t)\left(\frac{\partial\varphi}{\partial t}+a^2\Delta\varphi\right)dx\right)dt = -\lim_{\varepsilon\longrightarrow 0}\int_\varepsilon^\infty\left(\int_{\mathbb{R}^n}E(x,t)\left(\frac{\partial\varphi}{\partial t}+a^2\Delta\varphi\right)dx\right)dt.$$

Since $\operatorname{supp}\phi$ is compact, the integration by parts yields

$$-\int_{\varepsilon}^{\infty}\left(\int_{\mathbb{R}^n} E(x,t)\frac{\partial\varphi}{\partial t}dx\right)dt = \int_{\mathbb{R}^n} E(x,\varepsilon)\varphi(x,\varepsilon)dx + \int_{\varepsilon}^{\infty}\left(\int_{\mathbb{R}^n}\frac{\partial E}{\partial t}\varphi dx\right)dt.$$

Fix $R>0$ such that $\varphi(x,t)=0$ for $|x|\geq R/2$. Green's formula implies

$$\int_{\mathbb{R}^n} E(x,t)\Delta\varphi(x,t)dx = \int_{|x|\leq R} E(x,t)\Delta\varphi(x,t)dx = \int_{\mathbb{R}^n}(\Delta E(x,t))\varphi(x,t)dx,$$

since

$$\int_{|x|=R}\left(E\frac{\partial\varphi}{\partial\vec{n}} - \varphi\frac{\partial E}{\partial\vec{n}}\right)dx = 0$$

in view of the choice of R. Thus,

$$-\lim_{\varepsilon\longrightarrow 0}\int_{\varepsilon}^{\infty}\left(\int_{\mathbb{R}^n} E(x,t)\left(\frac{\partial\varphi}{\partial t}+a^2\Delta\varphi\right)dx\right)dt = \lim_{\varepsilon\longrightarrow 0}\left(\int E(x,\varepsilon)\varphi(x,\varepsilon)dx\right.$$
$$\left.+\int_{\varepsilon}^{\infty}\left(\int_{\mathbb{R}^n}(\frac{\partial E}{\partial t}-a^2\Delta E)\varphi dx\right)dt\right) = \lim_{\varepsilon\longrightarrow 0}\left(\int E(x,\varepsilon)\varphi(x,\varepsilon)dx\right),$$

where the last equality follows by (11). We need the following

Claim 1. *One has*

$$\lim_{\varepsilon\longrightarrow 0}\int E(x,\varepsilon)[\varphi(x,\varepsilon)-\varphi(x,0)]dx \longrightarrow 0.$$

For the proof, fix $R>0$ such that $\operatorname{supp}\phi\subset\{|(\mathrm{x},\mathrm{t})|<\mathrm{R}\}$. The function ϕ is Lipschitz continuous and hence, uniformly continuous on $\mathbb{R}^{n+1}$. Given $\alpha>0$ there exists $\delta>0$ such that $|\phi(x,\varepsilon)-\phi(x,0)|<\alpha/2$ for all $x\in\mathbb{R}^n$. Therefore,

$$\int E(x,\varepsilon)[\varphi(x,\varepsilon)-\varphi(x,0)]dx = I+II,$$

with

$$I=\int_{|x|<\delta} E(x,\varepsilon)[\varphi(x,\varepsilon)-\varphi(x,0)]dx,$$

and

$$II=\int_{\delta\leq|x|\leq R} E(x,\varepsilon)[\varphi(x,\varepsilon)-\varphi(x,0)]dx.$$

Then

$$|I|\leq(\alpha/2)\int_{\mathbb{R}^n} E(x,\varepsilon)dx = \alpha/2.$$

Set

$$M(\varepsilon)=\frac{1}{(2a\sqrt{\pi\varepsilon})^n}e^{-\frac{|\delta|^2}{4a^2\varepsilon}},$$

and $C=\sup_{x\in\mathbb{R}^n}|\phi(x)|$. Then $\sup_{|x|\geq\delta}E(x,\varepsilon)=M(\varepsilon)$ and

$$|II|\leq 2C\int_{\delta\leq|x|\leq R} E(x,\varepsilon)dx \leq 4CM(\varepsilon)R\to 0,\quad \varepsilon\to 0.$$

It follows that $|II|\leq\alpha/2$ for all ε small enough. This proves the claim.

We conclude that

$$\lim_{\varepsilon\longrightarrow 0}\int E(x,\varepsilon)\varphi(x,\varepsilon)dx = \lim_{\varepsilon\longrightarrow 0}(\int E(x,\varepsilon)\varphi(x,0)dx.$$

To finish the proof we need the following

Claim 2. *The following holds in $\mathcal{D}'(\mathbb{R}^n)$:*

$$\lim_{t\longrightarrow 0+} E(x,t) = \delta(x).$$

For the proof, let $\psi \in \mathcal{D}(\mathbb{R}^n)$. Since ψ has a compact support, there exists a constant $C > 0$ such that

$$|\psi(x) - \psi(0)| \le C|x|, x \in \mathbb{R}^n.$$

We have

$$\left| \int_{\mathbb{R}^n} E(x,t)(\psi(x) - \psi(0))dx \right| \le \frac{C}{(4\pi a^2 t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4a^2 t}} |x| dx.$$

Evaluating the last integral in the spherical coordinates (we denote by σ_n the surface of the unit sphere in $\mathbb{R}^n$) we obtain that the last integral is equal to

$$\frac{C\sigma_n}{(4\pi a^2 t)^{n/2}} \int_0^\infty e^{-\frac{r^2}{4a^2 t}} r^n dr = \frac{2C\sigma_n \sqrt{ta}}{\pi^{n/2}} \int_0^\infty e^{-u^2} u^n du = C'\sqrt{t}.$$

Hence,

$$\left| \int_{\mathbb{R}^n} E(x,t)(\psi(x) - \psi(0))dx \right| \longrightarrow 0, \text{ as } t \longrightarrow 0+.$$

Then, using (9), we have

$$\langle E(x,t), \psi \rangle = \int_{\mathbb{R}^n} E(x,t)\psi(x)dx = \psi(0) \int E(x,t)dx + \int E(x,t)(\psi(x) - \psi(0))dx$$
$$\longrightarrow \psi(0) = \langle \delta(x), \psi \rangle.$$

This proves the claim.

Let $\psi(x) = \varphi(x,0) \in \mathcal{D}(\mathbb{R}^n)$. Then

$$\langle \frac{\partial E}{\partial t} - a^2 \Delta E, \varphi \rangle = \lim_{\varepsilon \longrightarrow 0} \left(\int E(x,\varepsilon)\varphi(x,0)dx \right) = \varphi(0) = \langle \delta(x,t), \varphi \rangle.$$

This concludes the proof of (10).